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UNITED STATES PATENT APPLICATION FOR

**METHOD OF ANALYZING AND COMPUTING**

**ANISOTROPIC TURBULENT FLOWS**

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## **BACKGROUND OF THE INVENTION**

### **1. RELATED APPLICATION**

This application claims priority from co-pending provisional application Serial No. 60/152,391 filed September 3, 1999.

### **5 2. FIELD OF THE INVENTION**

This invention relates generally to the field of fluid dynamics. More particularly, the invention is a method for accurately and efficiently analyzing anisotropic turbulent flows of both gases and liquids.

### **3. BACKGROUND**

10 The study and engineering application of hydraulics or fluid mechanics dates to the dawn of civilization. The first irrigation canals were constructed before 5,000 BC. Theoretical analysis of fluid motions began in the modern scientific sense with Newton's *Principia* (1687) and his laws of motion. Bernoulli (1738) established a relationship between the pressure and velocity of an ideal fluid which became his famous theorem. d'Alembert (1744) deduced that the steady flow of an  
15 ideal fluid about a body produces no drag (d'Alembert's Paradox). Euler (1755) founded modern fluid mechanic analyses by deriving the differential equations of motion (continuity and momentum conservation) for an ideal (inviscid) isotropic fluid. Navier (1822) and Stokes (1845) independently derived the viscous terms necessary to extend Euler's equations to isotropic, viscous fluids. The equation of state and the conservation of energy equation (First Law of  
20 Thermodynamics) written utilizing Fourier's (1822) law of thermal energy transfer completed the classical Navier Stokes equations governing isotropic fluid motions.

Reynolds (1883) experimentally showed that there are two modes of fluid motion, laminar and turbulent motion, which occurs at large values of the dimensionless Reynolds number. Reynolds (1894), by considering a time average motion and a time dependent turbulent motion,



showed that turbulent motions introduce additional turbulent (Reynolds) stresses into the Navier Stokes equations, greatly increasing the number of flow properties required to describe turbulent flows.

5 Despite over a century of intense aircraft and aerospace developments, wind tunnel and laboratory experiments, theoretical analyses and numerical computations based on the Navier Stokes equations, the determination of a general theory of turbulent motion remains one of the last great unsolved problems of classical physics.

Maxwell (1858) founded modern kinetic theory when he introduced his new concept of the molecule velocity distribution function and deduced its equilibrium form. Maxwell (1867) derived the Maxwell transport equations and showed that the collision change integrals could be solved analytically without knowing the molecule velocity distribution function for Maxwell molecules. Boltzmann (1872) derived his integro-differential equation for Maxwell's molecular velocity distribution function and solved for his famous H-theorem, which proved that Maxwell's equilibrium form was correct. Chapman (1916), using Maxwell's transport equations, determined accurate general formulae for the gas transport coefficients. Enskog (1917) gave a general solution method for the Boltzmann equation, which showed that the Euler equations were the first and the Navier Stokes equations were the second approximate solutions to the Boltzmann equation and gave gas transport coefficients identical to Chapman's. Both the Chapman and Enskog analyses were for isotropic perfect gases, being based on perturbations of Maxwell's isotropic equilibrium solution. Their independent analyses cemented the belief in both the scientific and particularly the engineering professions that the Navier Stokes equations were generally and universally valid and that the correct way of analyzing both laminar and turbulent flows was through the Navier Stokes equations. This belief has been engraved in stone in both the scientific and engineering literature throughout the 20th century.

25 Burnett (1935) derived the third approximate solution to the Boltzmann equations using Enskog's solution method. Grad (1949) derived a thirteen moment method of solving



Boltzmann's equation which closely followed Enskog's methods. Both of these analyses have been proven to lack generality and have not been significant advances of the Navier Stokes theory. Bird (1963) introduced the Direct Simulation Monte Carlo (DSMC) numerical method for calculating rarified gas flows. Yen (1966) showed that the directional thermal energies

5 (temperatures) were vastly different in shock waves and that the longitudinal temperature overshoot its downstream value for shock Mach numbers greater than  $\sqrt{\frac{9}{5}}$  (1.34). Elliot and Baganoff (1974) showed that the Navier Stokes normal stress relationship and the Fourier energy flux component ratio were valid only in sound waves and were invalid at the shock end points. Elliot (1975) showed that the Navier Stokes normal stress relationship was invalid everywhere in shock

10 waves and incompatible with the directional thermal kinetic energy moments of the Boltzmann equation. DSMC numerical shock calculations confirmed all of these analytical predictions and numerically illustrated the anisotropic not isotropic fluid features of many simple gas flows.

Kliegel (1990), following a suggestion of Maxwell (1867) that near equilibrium flows were anisotropic and should be represented by an anisotropic, not isotropic Maxwellian, performed the

15 first general anisotropic solution analysis of the Boltzmann equation. Kliegel (1990) showed that the Euler-Navier Stokes-Burnett equation sequence was not the correct approximate solution sequence for the Boltzmann equation. He showed that the correct gas dynamic equation set was an anisotropic fluid seven equation set for the density, three fluid velocity components and three directional thermal kinetic energies (temperatures), not the classic isotropic fluid five equation

20 Euler-Navier Stokes set for the density, three fluid velocity components and total temperature. He also gave shear and directional energy flux relationship having the correct Mach number dependence for both sound waves and weak shocks. Kliegel's anisotropic equations resolved d'Alembert's paradox giving a profile pressure drag associated with pushing an ideal (shearless, energy fluxless) fluid about a body. They also correctly predicted the directional thermal energy

25 separations and overshoots and the thermal energy flux component changes occurring in shock



waves as predicted by Yen (1966) and numerical DSMC shock calculations. Bird (1994) summarizes recent DSMC calculational capabilities. Chen and Jaw (1998) presented a recent summary of classical isotropic fluid Navier Stokes based turbulent flow modelling.

5 The present disclosure correctly consolidates all previous fluid dynamic and kinetic theory analyses and extends it to define a new method of analyzing and computing anisotropic turbulent flows.

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## SUMMARY OF THE INVENTION

This disclosure concerns a simple method of deriving the correct kinetic anisotropic fluid turbulent flow equations from the Boltzmann equation and the correct thermal and turbulent moment closures to yield a new, closed set of fluid dynamic equations describing anisotropic  
5 turbulent motions, solving the century old quest for the solution to this classical problem.

Current turbulent flow analyses are based on the isotropic Navier Stokes equations, turbulent fluctuation moments of these equations and modeling of the unknown turbulent moments, identifying introduced unknown modeling coefficients by experiment. This results in a variety of semi-empirical turbulent flow analyses based on different Navier Stokes moment relationships satisfying mass,  
10 momentum and total energy conservation for an isotropic fluid, calibrated to approximately match a limited number of experiments. The limitations of current Navier Stokes based turbulent analyses are well known and appear to be irresolvable after a century of intense analytical and experimental effort.

The present invention provides a general, closed, anisotropic kinetic turbulence theory for gases and liquids based on new solutions of the Maxwell moment equations of the Boltzmann equations.  
15 These solutions provide a closed initial equation set for the four time average fluid mechanic variables, the sixteen time average thermal motion correlation and the sixteen time average turbulent motion correlations listed in Table I. Higher order closure sets involving higher order (fourth and fifth, etc.) thermal and turbulent moment sets are easily generated allowing solutions of turbulent flow problems to higher and higher accuracy if desired or needed for special problems. These higher order sets will  
20 not be further discussed since they add only unnecessary detail to the disclosure and reveal little that is new.

An algebraic differential equation solution method described in my co-pending application entitled "Method for Algebraically Solving Differential Equations, including Stiff Equations, to High Accuracy" may be used to reduce the thirty six governing differential equations to the solution of as



few as six differential equations (continuity, momentum, total energy and turbulent energy) plus thirty algebraic relationships. The actual equation set solved depends on the problem and accuracy desired.

The presented analytic anisotropic turbulence theory is mathematically correct, involving no semi-empirical modeling. The even order higher order moment closures involved are general and correct to known accuracy. The analysis can be used to solve the infinite set of Maxwell moment equations to any level desired to verify the accuracy of the initial closure set.

The anisotropic kinetic turbulence theory presented allows the calculation of turbulent motions with the same accuracy and computer resources as current laminar flow calculations.

The new teachings of the present invention may be summarized as follows:

1. A closed turbulence theory may be simply derived directly from the Boltzmann equation without turbulent correlation modelling or experimentally determined modelling coefficients.
2. Turbulent flows are totally inviscid flows, unchanged by molecular collisions.
3. Turbulence is universal, having the same governing equations and the same physical structure in both gases and liquids because turbulence is an inviscid flow phenomena described entirely by inviscid convective flow equations, independent of collisions and molecular interaction laws.
4. The fact that time average turbulent flow correlation between turbulent flow fluctuations and density or thermal velocity correlations are zero because there is no physical (collisional) connection between these fluctuations, since both the turbulent flow fluctuations and density are conserved in collisions.
5. A method of analyzing the Maxwell moment equations to obtain a closed set of equations which describe time average turbulent flows to any desired accuracy level.



All higher moment equation above the closure level can be solved in even/odd sets of two, thus yielding solutions to all Maxwell moment equations with error of Burnett order or less.

- 5
6. Direct predictions of turbulent transition locations, turbulent energy growth rates, shears and other thermal and turbulent flow quantities of physical importance can now be derived or calculated from solution of the presented equations with the same accuracy and computer resources as current laminar flow analyses or calculations.
- 10
7. Universal even order thermal and turbulent moment closure relationships exist and are explicitly given, allowing the termination and solution of the Maxwell moment equations at any desired odd moment equation set, and solution of the Boltzmann equation to any desired approximation.



## DETAILED DESCRIPTION OF THE INVENTION

In the following description, for purposes of explanation and not limitation, specific details are set forth in order to provide a thorough understanding of the present invention. However, it will be apparent to one skilled in the art that the present invention may be practiced in other  
5   embodiments that depart from these specific details. In other instances, detailed descriptions of well-known methods and devices are omitted so as to not obscure the description of the present invention with unnecessary detail.

The Boltzmann equation may be written in fluid dynamic collision rate form: [See Appendix, Equation set 1];

10   where  $\frac{D}{Dt} = [\rho F(v)]$  is the Lagrangian convective motion derivative,  $\rho$  is the fluid density,  $v$  is the molecular collision rate,  $F(v)$  is the molecular velocity ( $v$ ) distribution function and  $F(v)^*$  is the molecular velocity distribution function collided into  $F(v)$ . There is no approximation in writing the Boltzmann collision integral in this format, since the required thermal velocity moments of  $F(v)^*$  are correctly calculated from the full Boltzmann collision integral (Kliegel (1990)). The molecular  
15   collision rate is

$$v = \rho K$$

for Maxwell molecules, where  $K$  is a molecular collision constant.

The illustrative derivation will be done for Maxwell molecules for simplicity, since the collision change integrals are known exactly for Maxwell molecules. The same derivation can be performed for any molecular model, it is just more algebraically complex and adds nothing to the  
20   teachings.

Both the original Enskog (1917) solution as a perturbation about the equilibrium Maxwellian and the Kliegel (1990) solution as a perturbation about a non-equilibrium anisotropic



Maxwellian are Navier Stokes order accurate ( $\frac{1}{v^2}$  Burnett order error terms) and do not depend on density gradients. Thus, density gradient corrections to the molecular velocity distribution function and its moments belong to the Burnett order of corrections. This important fact, although present in the result of both analyses, is explicitly stated by neither author nor utilized to further extend or

5 simplify the analyses.

Multiplying the Boltzmann equation by the molecular velocity product  $v_1^l v_2^m v_3^n$ , where  $l, m, n$  are integers, and integrating over all possible velocities (indicated by the bar) yields the Maxwell molecular velocity moment equation: [See Appendix, Equation Set 2]. Derivation of the anisotropic turbulent flow equations from this equation will be discussed later.

10 Consider the molecular velocity ( $v$ ) to be composed of a mass mean fluid flow velocity ( $u$ ) and a thermal motion velocity ( $c$ ), then the Cartesian components of these velocities are related by

$$v_i = u_i + c_i$$

Multiplying the Boltzmann equation by the thermal velocity product  $c_1^l c_2^m c_3^n$ , where  $l, m$  and  $n$  are integers, and integrating over all possible thermal velocities (indicated by the bar) yields the

15 Maxwell thermal velocity moment equation: [See Appendix, Equation Set 3].

For odd moment ( $l + m + n = 2N + 1$ ,  $N$  integer) equations, the coefficients of the density gradient terms are even moments and products of lower even moments, which have equilibrium values. Thus these even moment coefficients must be zero to Burnett order for the density gradient terms to be of Burnett order. This requirement yields the general even order (fourth, sixth and

20 higher) density gradient independent thermal moment closure relationships: [See Appendix, Equation Set 4].

The fourth order thermal moment closure relationships can also obtained from Kliegel (1990) by simple calculation but he did not state these results nor utilize them in any manner.



Substituting these relationships into Equation Set 3, one obtains the closed odd order (third, fifth and higher) density gradient independent thermal moment equation. [See Appendix, Equation Set 5].

5 The classical Maxwell thermal velocity twenty moment equation set is: [See Appendix, Equation Set 6].

After using the fourth order moment density gradient independent closure relationships: [See Appendix, Equation Set 7], the closed directional thermal energy flux and structure equations become: [See Appendix, Equation Set 8] where the fourth order thermal moments have been replaced by products of second order moments through the fourth order density gradient  
10 independent closure relationships.

The above set of equations are the correct anisotropic gas dynamic flow equations with neglected terms of Burnett order in the convective terms of the closed thermal energy flux and structure equations. All other equations in the set are exact. These equations resolve the known deficiencies of the Euler-Navier Stokes-Burnett equation sets, d'Alembert's paradox, the normal stress incompatibility, the incorrect Mach number dependency of the Navier Stokes normal stress  
15 and Fourier energy flux components, and the incorrect coordinate rotation variance of the Burnett equations. These equations are valid for both sound waves and weak shocks, and thus will accurately predict all flows except in those regions of high change rates in distances comparable to the mean free path length, such as occur in strong shock waves and at boundaries in very rarified  
20 flows.

The twenty moment equations may be further simplified by performing an Enskog-type ordering analysis on all terms to obtain the reduced equation set: [See Appendix, Equation Set 9]



One can extend the above analysis to time average turbulent flows by considering the mass mean fluid flow velocity ( $u$ ) to be composed of a time average velocity ( $\bar{u}$ ), and a turbulent motion velocity ( $u'$ ), like Reynolds (1894). The Cartesian components of these velocities are related by:

$$u_i = \bar{u}_i + u'_i$$

5 where now

$$v_i = \bar{u}_i + u'_i + c_i$$

The turbulent motion ( $u'$ ) is inviscid, unaffected by collisions since the mean flow velocity ( $u$ ) is conserved in collisions and only the thermal ( $c$ ) velocities are changed in collisions. This important fact has not been previously recognized, explicitly stated or utilized in the scientific/engineering literature. Since both the density and turbulent motion are preserved in collisions, there is no collisional (physical) coupling between turbulent motion fluctuations, density fluctuations and thermal velocity correlation fluctuations. Thus, there is no time average correlation between these fluctuations. The averaging time required for the time average correlations between turbulent velocity fluctuations and molecular (number density and thermal motion correlation) fluctuations to be essentially zero is quite short, since the molecular fluctuations occur on a collisional time scale of nanoseconds and turbulent flow fluctuations occur on much larger fluid dynamic time scales (typically measured in milliseconds or greater). Thus, the time dependent time average anisotropic turbulent flow equations presented can be used to compute turbulent flows changing on fluid dynamic time scales for the vast majority of engineering flows. The time change adequacy of the time average anisotropic turbulent flow equations needs to be verified for extremely rapid flow changes such as those associated with shocks, flames and deteriorations. The only turbulent flow variables necessary to analyze anisotropic turbulent flows in the vast majority of physical cases are those given in Table I below, all other time average correlations being zero.



Table I

## Turbulent Flow Variables

Fluid Dynamic		Thermal Moments		Turbulent Moments	
$\bar{\rho}$	(1)	$\overline{c_1^2}$	(3)	$\overline{u_1'^2}$	(3)
$\bar{u}_1$	(3)	$\overline{c_1 c_2}$	(3)	$\overline{u_1' u_2'}$	(3)
		$\overline{c_1 c_1^2}$	(3)	$\overline{u_1' u_1'^2}$	(3)
		$\overline{c_1 c_2^2}$	(6)	$\overline{u_1' u_2'^2}$	(6)
		$\overline{c_1 c_2 c_3}$	(1)	$\overline{u_1' u_2' u_3'}$	(1)
Four Fluid Dynamic Variables		Sixteen Thermal Moment Variables		Sixteen Turbulent Moment Variables	

- The time average turbulent flow variables are the flow density ( $\bar{\rho}$ ), where the bar indicates time averaging, the three mass mean velocity components ( $\bar{u}_1$ ), the three anisotropic thermal kinetic energies ( $\overline{c_1^2}$ , where the two bars indicate both time averaging and averaging over all thermal velocities), the three thermal shear components ( $\overline{c_1 c_2}$ ), the nine thermal kinetic energy fluxes ( $\overline{c_1 c_1^2}$  and  $\overline{c_1 c_2^2}$ ), the thermal structure correlation ( $\overline{c_1 c_2 c_3}$ ), the three anisotropic turbulent kinetic energies ( $\overline{u_1'^2}$ ), the three turbulent shear components ( $\overline{u_1' u_2'}$ ), the nine turbulent kinetic energy fluxes ( $\overline{u_1' u_1'^2}$  and  $\overline{u_1' u_2'^2}$ ) and the turbulent structure correlation ( $\overline{u_1' u_2' u_3'}$ ).

- The instantaneous Maxwell molecular velocity ( $v$ ) and the thermal velocity ( $c$ ) moment equations (Equation Sets 2 and 3) are both valid at every instant of time and the instantaneous Maxwell thermal velocity moment equations can be derived from the instantaneous Maxwell molecular velocity moment equations. The time average Maxwell molecular velocity ( $v$ ) and time average thermal velocity ( $c$ ) moment equations are also both valid simultaneously. The time average thermal velocity equations yield equations governing the behavior of the anisotropic time



average thermal directional kinetic energies, shears, directional kinetic energy fluxes and structure correlations: [See Appendix, Equation Set 10].

The time average molecular velocity equations yield equations governing mass and momentum conservation and the behavior of both the time average thermal and turbulent  
5 directional energies, shears, energy fluxes and structure correlations: [See Appendix, Equations Set 11].

After eliminating the collision change terms with the time average thermal Maxwell moment equations and other simplifications, one obtains the equations governing the behavior of the anisotropic time average turbulent directional kinetic energies, shears, directional energy fluxes  
10 and structure correlations: [See Appendix, Equation Set 12].

Both sets of third order directional kinetic energy fluxes and structure correlation equations are closed using the fourth order density gradient independent time average thermal and turbulent moment closure relationships: [See Appendix, Equation Sets 13 and 14].

One thus obtains the complete closed Maxwell twenty moment equation set governing  
15 anisotropic time average turbulent flows: [See Appendix, Equation Set 15].

By the same process, corresponding higher order time average turbulent moment equation sets may be derived using higher order closure relationships. The general density gradient independent time average thermal and turbulent even moment closure relationships are: [See Appendix, Equation Sets 16 and 17].

20 These closures allow the infinite set of time average Maxwell moment equations governing turbulent flows to be closed at any desired odd order level starting at the third order. If one closes at the third order (say), then one can solve all higher order moment sets (fourth and fifth, six and seventh, etc.) in order, where the errors are still of Burnett order in the convective terms of the highest order moment equations (fifth, seventh, etc.) considered in the set. The method thus allows



solution of all the Maxwell moment equations, both those sets below and above the chosen closure level.

The above results may be simply extended in many ways. For example, the general even order density gradient independent moment closure relationships (Equation Set 4) are valid in equilibrium. Thus, [See Appendix, Equation Set 18] and the even order collision integrals

5  $\left( \overline{c_1^{l+1} c_2^m c_3^n} - \overline{c_1^{l+1} c_2^m c_3^n}, \text{etc.} \right)$  may be calculated for all molecular models.

It will be recognized that the above-described invention may be embodied in other specific forms without departing from the spirit or essential characteristics of the disclosure. Specifically, the anisotropic turbulent flow equations may be reformatted and/or approximated and computed in many different ways to solve specific problems. Obvious first steps would be to add gravitational

10 forces or to sum the directional turbulent energy equations to obtain a total turbulent energy equation or to reduce the complexity of the equation set through a Prandtl (1904) boundary layer analysis. The analysis presented (valid for monatomic perfect gases) can be expanded to diatomic (and polyatomic) gases, gas mixtures, reacting gas mixtures, plasmas, etc. through known methods,

15 Chapman and Cowling (1970). The analysis can be extended to isotropic fluids by setting the three directional thermal kinetic energies  $\overline{c_1^2}$  or  $\overline{\overline{c_1^2}}$  equal, and solving only the summed total thermal energy equation, rather than the three directional thermal kinetic equations. Since this reduces the Equation Set 9 to the Navier Stokes equations, Kliegel (1990), the analysis can be extended to liquids by setting both the three directional thermal kinetic energies equal and the density constant.

20 The closed anisotropic turbulent flow equation set can be utilized with existing Reynolds averaged Navier Stokes equation solvers to solve incompressible isotropic fluid problems. Thus, it is understood that the invention is not to be limited by the foregoing illustrative details, but rather is to be defined by the appended claims.



While all equation sets may be solved by straightforward computation, they may be more efficiently solved using the method described in my co-pending application entitled "Method for Algebraically Solving Differential Equations, Including Stiff Equations, to High Accuracy".

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## APPENDIX

Equation Set 1  
(1-1)

$$\begin{aligned}\frac{D}{Dt}[\rho F(v)] &= \frac{\partial}{\partial t}[\rho F(v)] + v_1 \frac{\partial}{\partial x_1}[\rho F(v)] + v_2 \frac{\partial}{\partial x_2}[\rho F(v)] + v_3 \frac{\partial}{\partial x_3}[\rho F(v)] \\ &= \rho v [F(v)^* - F(v)]\end{aligned}$$

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Equation Set 2  
(2-1)

$$\begin{aligned} & \frac{\partial}{\partial t} \left[ \rho \overline{v_1^l v_2^m v_3^n} \right] + \frac{\partial}{\partial x_1} \left[ \rho \overline{v_1^{l+1} v_2^m v_3^n} \right] + \frac{\partial}{\partial x_2} \left[ \rho \overline{v_1^l v_2^{m+1} v_3^n} \right] + \frac{\partial}{\partial x_3} \left[ \rho \overline{v_1^l v_2^m v_3^{n+1}} \right] \\ &= \rho v \left[ \overline{v_1^l v_2^m v_3^n}^* - \overline{v_1^l v_2^m v_3^n} \right] \end{aligned}$$

where  $\overline{v_1^l v_2^m v_3^n}$  is the instantaneous molecular velocity moment  $v_1^l v_2^m v_3^n$  averaged over  $F(v)$  and  $\overline{v_1^l v_2^m v_3^n}^*$  is the instantaneous scattered molecular velocity moment  $v_1^l v_2^m v_3^n$  averaged over  $F(v)^*$ .

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Equation Set 3  
(3-1)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1^l c_2^m c_3^n}] + u_1 \frac{\partial}{\partial x_1} [\overline{c_1^l c_2^m c_3^n}] + u_2 \frac{\partial}{\partial x_2} [\overline{c_1^l c_2^m c_3^n}] + u_3 \frac{\partial}{\partial x_3} [\overline{c_1^l c_2^m c_3^n}] \\
 & - l \overline{c_1^{l-1} c_2^m c_3^n} \left[ \frac{\partial}{\partial x_1} [\overline{c_1^2}] + \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \frac{\partial}{\partial x_3} [\overline{c_1 c_3}] \right] \\
 & - m \overline{c_1^l c_2^{m-1} c_3^n} \left[ \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \frac{\partial}{\partial x_2} [\overline{c_2^2}] + \frac{\partial}{\partial x_3} [\overline{c_2 c_3}] \right] \\
 & - n \overline{c_1^l c_2^m c_3^{n-1}} \left[ \frac{\partial}{\partial x_1} [\overline{c_1 c_3}] + \frac{\partial}{\partial x_2} [\overline{c_2 c_3}] + \frac{\partial}{\partial x_3} [\overline{c_3^2}] \right] \\
 & + l \left[ \overline{c_1^l c_2^m c_3^n} \frac{\partial u_1}{\partial x_1} + \overline{c_1^{l-1} c_2^{m+1} c_3^n} \frac{\partial u_1}{\partial x_2} + \overline{c_1^{l-1} c_2^m c_3^{n+1}} \frac{\partial u_1}{\partial x_3} \right] \\
 & + m \left[ \overline{c_1^{l+1} c_2^{m-1} c_3^n} \frac{\partial u_2}{\partial x_1} + \overline{c_1^l c_2^m c_3^n} \frac{\partial u_2}{\partial x_2} + \overline{c_1^l c_2^{m-1} c_3^{n+1}} \frac{\partial u_2}{\partial x_3} \right] \\
 & + n \left[ \overline{c_1^{l+1} c_2^m c_3^{n-1}} \frac{\partial u_3}{\partial x_1} + \overline{c_1^l c_2^{m+1} c_3^{n-1}} \frac{\partial u_3}{\partial x_2} + \overline{c_1^l c_2^m c_3^n} \frac{\partial u_3}{\partial x_3} \right] \\
 & + \frac{\partial}{\partial x_1} [\overline{c_1^{l+1} c_2^m c_3^n}] + \frac{\partial}{\partial x_2} [\overline{c_1^l c_2^{m+1} c_3^n}] + \frac{\partial}{\partial x_3} [\overline{c_1^l c_2^m c_3^{n+1}}] \\
 & + \left[ \overline{c_1^{l+1} c_2^m c_3^n} - l (\overline{c_1^2}) \overline{c_1^{l-1} c_2^m c_3^n} - m [\overline{c_1 c_2}] \overline{c_1^l c_2^{m-1} c_3^n} - n [\overline{c_1 c_3}] \overline{c_1^l c_2^m c_3^{n-1}} \right] \frac{1}{\rho} \frac{\partial \rho}{\partial x_1} \\
 & + \left[ \overline{c_1^l c_2^{m+1} c_3^n} - l (\overline{c_1 c_2}) \overline{c_1^{l-1} c_2^m c_3^n} - m [\overline{c_2^2}] \overline{c_1^l c_2^{m-1} c_3^n} - n [\overline{c_2 c_3}] \overline{c_1^l c_2^m c_3^{n-1}} \right] \frac{1}{\rho} \frac{\partial \rho}{\partial x_2} \\
 & + \left[ \overline{c_1^l c_2^m c_3^{n+1}} - l [\overline{c_1 c_3}] \overline{c_1^{l-1} c_2^m c_3^n} - m [\overline{c_2 c_3}] \overline{c_1^l c_2^{m-1} c_3^n} - n [\overline{c_3^2}] \overline{c_1^l c_2^m c_3^{n-1}} \right] \frac{1}{\rho} \frac{\partial \rho}{\partial x_3} \\
 & = v \left[ \overline{c_1^l c_2^m c_3^n}^* - \overline{c_1^l c_2^m c_3^n} \right]
 \end{aligned}$$

where  $\overline{c_1^l c_2^m c_3^n}$  is the instantaneous thermal velocity moment  $c_1^l c_2^m c_3^n$  averaged over  $F(v)$ , and

$\overline{c_1^l c_2^m c_3^n}^*$  is the instantaneous scattered thermal velocity moment  $c_1^l c_2^m c_3^n$  averaged over  $F(v)^*$ . The

latter may be calculated from the full Boltzmann collision change integrals, Kliegel (1990).



# Equation Set 4

(4-1); (4-2); (4-3)

$$\overline{c_1^{l+1} c_2^m c_3^n} = l \left[ \overline{c_1^2} \right] \overline{c_1^{l-1} c_2^m c_3^n} + m \left[ \overline{c_1 c_2} \right] \overline{c_1^l c_2^{m-1} c_3^n} + n \left[ \overline{c_1 c_3} \right] \overline{c_1^l c_2^m c_3^{n-1}}$$

$$\overline{c_1^l c_2^{m+1} c_3^n} = l \left[ \overline{c_1 c_2} \right] \overline{c_1^{l-1} c_2^m c_3^n} + m \left[ \overline{c_2^2} \right] \overline{c_1^l c_2^{m-1} c_3^n} + n \left[ \overline{c_2 c_3} \right] \overline{c_1^l c_2^m c_3^{n-1}}$$

$$\overline{c_1^l c_2^m c_3^{n+1}} = l \left[ \overline{c_1 c_3} \right] \overline{c_1^{l-1} c_2^m c_3^n} + m \left[ \overline{c_2 c_3} \right] \overline{c_1^l c_2^{m-1} c_3^n} + n \left[ \overline{c_3^2} \right] \overline{c_1^l c_2^m c_3^{n-1}}$$

neglected terms being of Burnett order, where  $l+m+n = 2N + 1$ , an odd number.

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Equation Set 5  
(5-1)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1^l c_2^m c_3^n}] + u_1 \frac{\partial}{\partial x_1} [\overline{c_1^l c_2^m c_3^n}] + u_2 \frac{\partial}{\partial x_2} [\overline{c_1^l c_2^m c_3^n}] + u_3 \frac{\partial}{\partial x_3} [\overline{c_1^l c_2^m c_3^n}] \\
 & + l \left[ \overline{c_1^l c_2^m c_3^n} \frac{\partial u_1}{\partial x_1} + \overline{c_1^{l-1} c_2^{m+1} c_3^n} \frac{\partial u_1}{\partial x_2} + \overline{c_1^{l-1} c_2^m c_3^{n+1}} \frac{\partial u_1}{\partial x_3} + \overline{c_1^2} \frac{\partial}{\partial x_1} [\overline{c_1^{l-1} c_2^m c_3^n}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{c_1^{l-1} c_2^m c_3^n}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{c_1^{l-1} c_2^m c_3^n}] \right] \\
 & + m \left[ \overline{c_1^{l+1} c_2^{m-1} c_3^n} \frac{\partial u_2}{\partial x_1} + \overline{c_1^l c_2^m c_3^n} \frac{\partial u_2}{\partial x_2} + \overline{c_1^l c_2^{m-1} c_3^{n+1}} \frac{\partial u_2}{\partial x_3} + \overline{c_1 c_2} \frac{\partial}{\partial x_1} [\overline{c_1^l c_2^{m-1} c_3^n}] + \overline{c_2^2} \frac{\partial}{\partial x_2} [\overline{c_1^l c_2^{m-1} c_3^n}] + \overline{c_2 c_3} \frac{\partial}{\partial x_3} [\overline{c_1^l c_2^{m-1} c_3^n}] \right] \\
 & + n \left[ \overline{c_1^{l+1} c_2^m c_3^{n-1}} \frac{\partial u_3}{\partial x_1} + \overline{c_1^l c_2^{m+1} c_3^{n-1}} \frac{\partial u_3}{\partial x_2} + \overline{c_1^l c_2^m c_3^n} \frac{\partial u_3}{\partial x_3} + \overline{c_1 c_3} \frac{\partial}{\partial x_1} [\overline{c_1^l c_2^m c_3^{n-1}}] + \overline{c_2 c_3} \frac{\partial}{\partial x_2} [\overline{c_1^l c_2^m c_3^{n-1}}] + \overline{c_3^2} \frac{\partial}{\partial x_3} [\overline{c_1^l c_2^m c_3^{n-1}}] \right] \\
 & = v \left[ \overline{c_1^l c_2^m c_3^n}^* - \overline{c_1^l c_2^m c_3^n} \right]
 \end{aligned}$$

where  $l+m+n = 2N + 1$ , an odd number and the neglected terms are of Burnett order in the convective terms (left hand side).



Equation Set 6  
(Twenty Moment Equations)

Continuity (6-1)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_1} [\rho u_1] + \frac{\partial}{\partial x_2} [\rho u_2] + \frac{\partial}{\partial x_3} [\rho u_3] = 0$$

Momentum (6-2)

$$\frac{\partial}{\partial t} [\rho u_1] + \frac{\partial}{\partial x_1} [\rho [u_1^2 + \overline{c_1^2}]] + \frac{\partial}{\partial x_2} [\rho [u_1 u_2 + \overline{c_1 c_2}]] + \frac{\partial}{\partial x_3} [\rho [u_1 u_3 + \overline{c_1 c_3}]] = 0$$

Directional Thermal Energy (6-3)

$$\begin{aligned} & \frac{\partial}{\partial t} [\overline{c_1^2}] + u_1 \frac{\partial}{\partial x_1} [\overline{c_1^2}] + u_2 \frac{\partial}{\partial x_2} [\overline{c_1^2}] + u_3 \frac{\partial}{\partial x_3} [\overline{c_1^2}] \\ & + 2 \left[ \overline{c_1^2} \frac{\partial u_1}{\partial x_1} + \overline{c_1 c_2} \frac{\partial u_1}{\partial x_2} + \overline{c_1 c_3} \frac{\partial u_1}{\partial x_3} \right] \\ & + \frac{1}{\rho} \left[ \frac{\partial}{\partial x_1} [\rho \overline{c_1 c_1^2}] + \frac{\partial}{\partial x_2} [\rho \overline{c_2 c_1^2}] + \frac{\partial}{\partial x_3} [\rho \overline{c_3 c_1^2}] \right] \\ & = v \left[ \frac{1}{3} \overline{c^2} - \overline{c_1^2} \right] \end{aligned}$$

Thermal Shear (6-4)

$$\begin{aligned} & \frac{\partial}{\partial t} [\overline{c_1 c_2}] + u_1 \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + u_2 \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + u_3 \frac{\partial}{\partial x_3} [\overline{c_1 c_2}] \\ & + \overline{c_1 c_2} \frac{\partial u_1}{\partial x_1} + \overline{c_2^2} \frac{\partial u_1}{\partial x_2} + \overline{c_2 c_3} \frac{\partial u_1}{\partial x_3} \\ & + \overline{c_1^2} \frac{\partial u_2}{\partial x_1} + \overline{c_1 c_2} \frac{\partial u_2}{\partial x_2} + \overline{c_1 c_3} \frac{\partial u_2}{\partial x_3} \\ & + \frac{1}{\rho} \left[ \frac{\partial}{\partial x_1} [\rho \overline{c_2 c_1^2}] + \frac{\partial}{\partial x_2} [\rho \overline{c_1 c_2^2}] + \frac{\partial}{\partial x_3} [\rho \overline{c_1 c_2 c_3}] \right] \\ & = -v \overline{c_1 c_2} \end{aligned}$$



# Equation Set 6

## Directional Thermal Energy Fluxes (6-5)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1 c_1^2}] + u_1 \frac{\partial}{\partial x_1} [\overline{c_1 c_1^2}] + u_2 \frac{\partial}{\partial x_2} [\overline{c_1 c_1^2}] + u_3 \frac{\partial}{\partial x_3} [\overline{c_1 c_1^2}] \\
 & + 3 \left[ \overline{c_1 c_1^2} \frac{\partial u_1}{\partial x_1} + \overline{c_2 c_1^2} \frac{\partial u_1}{\partial x_2} + \overline{c_3 c_1^2} \frac{\partial u_1}{\partial x_3} \right] \\
 & - 3 \overline{c_1^2} \left[ \frac{\partial}{\partial x_1} [\overline{c_1^2}] + \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \frac{\partial}{\partial x_3} [\overline{c_1 c_3}] \right] \\
 & + \frac{\partial}{\partial x_1} [\overline{c_1^2 c_1^2}] + \frac{\partial}{\partial x_2} [\overline{c_1 c_2 c_1^2}] + \frac{\partial}{\partial x_3} [\overline{c_1 c_3 c_1^2}] \\
 & + \left[ \left[ \overline{c_1^2 c_1^2} - 3 \overline{c_1^2} [\overline{c_1^2}] \right] \frac{1}{\rho} \frac{\partial \rho}{\partial x_1} + \left[ \overline{c_1 c_2 c_1^2} - 3 \overline{c_1 c_2} [\overline{c_1^2}] \right] \frac{1}{\rho} \frac{\partial \rho}{\partial x_2} + \left[ \overline{c_1 c_3 c_1^2} - 3 \overline{c_1 c_3} [\overline{c_1^2}] \right] \frac{1}{\rho} \frac{\partial \rho}{\partial x_3} \right] \\
 & = \frac{v}{2} [\overline{c_1 c^2} - 3 \overline{c_1 c_1^2}]
 \end{aligned}$$



(6-6)

$$\begin{aligned}
& \frac{\partial}{\partial t} [\overline{c_1 c_2^2}] + u_1 \frac{\partial}{\partial x_1} [\overline{c_1 c_2^2}] + u_2 \frac{\partial}{\partial x_2} [\overline{c_1 c_2^2}] + u_3 \frac{\partial}{\partial x_3} [\overline{c_1 c_2^2}] \\
& + \overline{c_1 c_2^2} \frac{\partial u_1}{\partial x_1} + \overline{c_2 c_2^2} \frac{\partial u_1}{\partial x_2} + \overline{c_3 c_2^2} \frac{\partial u_1}{\partial x_3} \\
& + 2 \left[ \overline{c_2 c_1^2} \frac{\partial u_2}{\partial x_1} + \overline{c_1 c_2^2} \frac{\partial u_2}{\partial x_2} + \overline{c_1 c_2 c_3} \frac{\partial u_2}{\partial x_3} \right] \\
& - \overline{c_2^2} \left[ \frac{\partial}{\partial x_1} [\overline{c_1^2}] + \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \frac{\partial}{\partial x_3} [\overline{c_1 c_3}] \right] \\
& - 2 \overline{c_1 c_2} \left[ \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \frac{\partial}{\partial x_2} [\overline{c_2^2}] + \frac{\partial}{\partial x_3} [\overline{c_2 c_3}] \right] \\
& + \frac{\partial}{\partial x_1} [\overline{c_1^2 c_2^2}] + \frac{\partial}{\partial x_2} [\overline{c_1 c_2 c_2^2}] + \frac{\partial}{\partial x_3} [\overline{c_1 c_3 c_2^2}] \\
& + [\overline{c_1^2 c_2^2} - \overline{c_1^2} [\overline{c_2^2}] - 2 \overline{c_1 c_2} [\overline{c_1 c_2}]] \frac{1}{\rho} \frac{\partial \rho}{\partial x_1} \\
& + [\overline{c_1 c_2 c_2^2} - 3 \overline{c_1 c_2} [\overline{c_2^2}]] \frac{1}{\rho} \frac{\partial \rho}{\partial x_2} \\
& + [\overline{c_1 c_3 c_2^2} - \overline{c_1 c_3} [\overline{c_2^2}] - 2 \overline{c_1 c_2} [\overline{c_2 c_3}]] \frac{1}{\rho} \frac{\partial \rho}{\partial x_3} \\
& = \frac{\nu}{2} \left[ \frac{1}{3} \overline{c_2 c^2} - 3 \overline{c_2 c_1^2} \right]
\end{aligned}$$



# Equation Set 6

## Thermal Structure (6-7)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1 c_2 c_3}] + u_1 \frac{\partial}{\partial x_1} [\overline{c_1 c_2 c_3}] + u_2 \frac{\partial}{\partial x_2} [\overline{c_1 c_2 c_3}] + u_3 \frac{\partial}{\partial x_3} [\overline{c_1 c_2 c_3}] \\
 & + \overline{c_1 c_2 c_3} \frac{\partial u_1}{\partial x_1} + \overline{c_3 c_2^2} \frac{\partial u_1}{\partial x_2} + \overline{c_2 c_3^2} \frac{\partial u_1}{\partial x_3} \\
 & + \overline{c_3 c_1^2} \frac{\partial u_2}{\partial x_1} + \overline{c_1 c_2 c_3} \frac{\partial u_2}{\partial x_2} + \overline{c_1 c_3^2} \frac{\partial u_2}{\partial x_3} \\
 & + \overline{c_2 c_1^2} \frac{\partial u_3}{\partial x_1} + \overline{c_1 c_2^2} \frac{\partial u_3}{\partial x_2} + \overline{c_1 c_2 c_3} \frac{\partial u_3}{\partial x_3} \\
 & - \overline{c_2 c_3} \left[ \frac{\partial}{\partial x_1} [\overline{c_1^2}] + \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \frac{\partial}{\partial x_3} [\overline{c_1 c_3}] \right] \\
 & - \overline{c_1 c_3} \left[ \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \frac{\partial}{\partial x_2} [\overline{c_2^2}] + \frac{\partial}{\partial x_3} [\overline{c_2 c_3}] \right] \\
 & - \overline{c_1 c_2} \left[ \frac{\partial}{\partial x_1} [\overline{c_1 c_3}] + \frac{\partial}{\partial x_2} [\overline{c_2 c_3}] + \frac{\partial}{\partial x_3} [\overline{c_3^2}] \right] \\
 & + \frac{\partial}{\partial x_1} [\overline{c_2 c_3 c_1^2}] + \frac{\partial}{\partial x_2} [\overline{c_1 c_3 c_2^2}] + \frac{\partial}{\partial x_3} [\overline{c_1 c_2 c_3^2}] \\
 & + \left[ \overline{c_2 c_3 c_1^2} - \overline{c_2 c_3} [\overline{c_1^2}] - 2 \overline{c_1 c_2} [\overline{c_1 c_3}] \right] \frac{1}{\rho} \frac{\partial \rho}{\partial x_1} \\
 & + \left[ \overline{c_1 c_3 c_2^2} - \overline{c_1 c_3} [\overline{c_2^2}] - 2 \overline{c_1 c_2} [\overline{c_2 c_3}] \right] \frac{1}{\rho} \frac{\partial \rho}{\partial x_2} \\
 & + \left[ \overline{c_1 c_2 c_3^2} - \overline{c_1 c_2} [\overline{c_3^2}] - 2 \overline{c_1 c_3} [\overline{c_2 c_3}] \right] \frac{1}{\rho} \frac{\partial \rho}{\partial x_3} \\
 & = -\frac{3}{2} v \overline{c_1 c_2 c_3}
 \end{aligned}$$

where  $\overline{c^2} = \overline{c_1^2} + \overline{c_2^2} + \overline{c_3^2}$  and the collision change terms (right hand sides) have been calculated using the method of Kliegel (1990).



Equation Set 7  
(7-1; 7-2; 7-3; 7-4)

$$\overline{c_1^2 c_1^2} = 3 \overline{c_1^2} [\overline{c_1^2}]$$

$$\overline{c_1 c_2 c_1^2} = 3 \overline{c_1 c_2} [\overline{c_1^2}]$$

$$\overline{c_1^2 c_2^2} = \overline{c_1^2} [\overline{c_2^2}] + 2 \overline{c_1 c_2} [\overline{c_1 c_2}]$$

$$\overline{c_1 c_2 c_3^2} = \overline{c_1 c_2} [\overline{c_3^2}] + 2 \overline{c_1 c_3} [\overline{c_2 c_3}]$$

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# Equation Set 8

## Closed Directional Thermal Energy Fluxes (8-1; 8-2)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1 c_1^2}] + u_1 \frac{\partial}{\partial x_1} [\overline{c_1 c_1^2}] + u_2 \frac{\partial}{\partial x_2} [\overline{c_1 c_1^2}] + u_3 \frac{\partial}{\partial x_3} [\overline{c_1 c_1^2}] \\
 & + 3 \left[ \overline{c_1 c_1^2} \frac{\partial u_1}{\partial x_1} + \overline{c_2 c_1^2} \frac{\partial u_1}{\partial x_2} + \overline{c_3 c_1^2} \frac{\partial u_1}{\partial x_3} \right] \\
 & + 3 \left[ \overline{c_1^2} \frac{\partial}{\partial x_1} [\overline{c_1^2}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{c_1^2}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{c_1^2}] \right] \\
 & = \frac{v}{2} [\overline{c_1 c^2} - 3 \overline{c_1 c_1^2}]
 \end{aligned}$$

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1 c_2^2}] + u_1 \frac{\partial}{\partial x_1} [\overline{c_1 c_2^2}] + u_2 \frac{\partial}{\partial x_2} [\overline{c_1 c_2^2}] + u_3 \frac{\partial}{\partial x_3} [\overline{c_1 c_2^2}] \\
 & + \overline{c_1 c_2^2} \frac{\partial u_1}{\partial x_1} + \overline{c_2 c_2^2} \frac{\partial u_1}{\partial x_2} + \overline{c_3 c_2^2} \frac{\partial u_1}{\partial x_3} \\
 & + 2 \left[ \overline{c_2 c_1^2} \frac{\partial u_2}{\partial x_1} + \overline{c_1 c_2^2} \frac{\partial u_2}{\partial x_2} + \overline{c_1 c_2 c_3} \frac{\partial u_2}{\partial x_3} \right] \\
 & + \overline{c_1^2} \frac{\partial}{\partial x_1} [\overline{c_2^2}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{c_2^2}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{c_2^2}] \\
 & + 2 \left[ \overline{c_1 c_2} \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \overline{c_2^2} \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \overline{c_2 c_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_2}] \right] \\
 & = \frac{v}{6} [\overline{c_1 c^2} - 9 \overline{c_1 c_2^2}]
 \end{aligned}$$



# Equation Set 8

## Closed Thermal Structure (8-3)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1 c_2 c_3}] + u_1 \frac{\partial}{\partial x_1} [\overline{c_1 c_2 c_3}] + u_2 \frac{\partial}{\partial x_2} [\overline{c_1 c_2 c_3}] + u_3 \frac{\partial}{\partial x_3} [\overline{c_1 c_2 c_3}] \\
 & + \overline{c_1 c_2 c_3} \frac{\partial u_1}{\partial x_1} + \overline{c_3 c_2^2} \frac{\partial u_1}{\partial x_2} + \overline{c_2 c_3^2} \frac{\partial u_1}{\partial x_3} \\
 & + \overline{c_3 c_1^2} \frac{\partial u_2}{\partial x_1} + \overline{c_1 c_2 c_3} \frac{\partial u_2}{\partial x_2} + \overline{c_1 c_3^2} \frac{\partial u_2}{\partial x_3} \\
 & + \overline{c_2 c_1^2} \frac{\partial u_3}{\partial x_1} + \overline{c_1 c_2^2} \frac{\partial u_3}{\partial x_2} + \overline{c_1 c_2 c_3} \frac{\partial u_3}{\partial x_3} \\
 & + \overline{c_1^2} \frac{\partial}{\partial x_1} [\overline{c_2 c_3}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{c_2 c_3}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{c_2 c_3}] \\
 & + \overline{c_1 c_2} \frac{\partial}{\partial x_1} [\overline{c_1 c_3}] + \overline{c_2^2} \frac{\partial}{\partial x_2} [\overline{c_1 c_3}] + \overline{c_2 c_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_3}] \\
 & + \overline{c_1 c_3} \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \overline{c_2 c_3} \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \overline{c_3^2} \frac{\partial}{\partial x_3} [\overline{c_1 c_2}] \\
 & = -\frac{3}{2} \nu \overline{c_1 c_2 c_3}
 \end{aligned}$$

the neglected terms are of Burnett order in the convective terms (left hand sides).



## Equation Set 9

Continuity (9-1)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_1} [\rho u_1] + \frac{\partial}{\partial x_2} [\rho u_2] + \frac{\partial}{\partial x_3} [\rho u_3] = 0$$

Momentum (9-2)

$$\begin{aligned} & \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_1}{\partial x_3} \\ & + \frac{1}{\rho} \left[ \frac{\partial}{\partial x_1} [\rho \overline{c_1^2}] + \frac{\partial}{\partial x_2} [\rho \overline{c_1 c_2}] + \frac{\partial}{\partial x_3} [\rho \overline{c_1 c_3}] \right] \\ & = 0 \end{aligned}$$

Directional Thermal Energy (9-3)

$$\begin{aligned} & \frac{\partial}{\partial t} [\overline{c_1^2}] + u_1 \frac{\partial}{\partial x_1} [\overline{c_1^2}] + u_2 \frac{\partial}{\partial x_2} [\overline{c_1^2}] + u_3 \frac{\partial}{\partial x_3} [\overline{c_1^2}] \\ & + 2 \left[ \overline{c_1^2} \frac{\partial u_1}{\partial x_1} + \overline{c_1 c_2} \frac{\partial u_1}{\partial x_2} + \overline{c_1 c_3} \frac{\partial u_1}{\partial x_3} \right] \\ & + \frac{1}{\rho} \left[ \frac{\partial}{\partial x_1} [\rho \overline{c_1 c_1^2}] + \frac{\partial}{\partial x_2} [\rho \overline{c_2 c_1^2}] + \frac{\partial}{\partial x_3} [\rho \overline{c_3 c_1^2}] \right] \\ & = v \left[ \frac{1}{3} \overline{c^2} - \overline{c_1^2} \right] \end{aligned}$$

Thermal Shear (9-4)

$$c_1 c_2 = -\frac{1}{v} \left[ \overline{c_2^2} \frac{\partial u_1}{\partial x_2} + \overline{c_1^2} \frac{\partial u_2}{\partial x_1} \right]$$

Directional Thermal Energy Fluxes (9-5)

$$\begin{aligned} \overline{c_1 c_1^2} &= -\frac{\overline{c_1^2}}{2 v} \left[ 7 \frac{\partial}{\partial x_1} [\overline{c_1^2}] + \frac{\partial}{\partial x_1} [\overline{c_2^2}] + \frac{\partial}{\partial x_1} [\overline{c_3^2}] \right] \\ \overline{c_2 c_1^2} &= -\frac{\overline{c_2^2}}{6 v} \left[ 5 \frac{\partial}{\partial x_2} [\overline{c_1^2}] + 3 \frac{\partial}{\partial x_2} [\overline{c_2^2}] + \frac{\partial}{\partial x_2} [\overline{c_3^2}] \right] \end{aligned}$$

Equation Set 9



	1970	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025	2026	2027	2028	2029	2030	2031	2032	2033	2034	2035	2036	2037	2038	2039	2040	2041	2042	2043	2044	2045	2046	2047	2048	2049	2050	2051	2052	2053	2054	2055	2056	2057	2058	2059	2060	2061	2062	2063	2064	2065	2066	2067	2068	2069	2070	2071	2072	2073	2074	2075	2076	2077	2078	2079	2080	2081	2082	2083	2084	2085	2086	2087	2088	2089	2090	2091	2092	2093	2094	2095	2096	2097	2098	2099	2100	2101	2102	2103	2104	2105	2106	2107	2108	2109	2110	2111	2112	2113	2114	2115	2116	2117	2118	2119	2120	2121	2122	2123	2124	2125	2126	2127	2128	2129	2130	2131	2132	2133	2134	2135	2136	2137	2138	2139	2140	2141	2142	2143	2144	2145	2146	2147	2148	2149	2150	2151	2152	2153	2154	2155	2156	2157	2158	2159	2160	2161	2162	2163	2164	2165	2166	2167	2168	2169	2170	2171	2172	2173	2174	2175	2176	2177	2178	2179	2180	2181	2182	2183	2184	2185	2186	2187	2188	2189	2190	2191	2192	2193	2194	2195	2196	2197	2198	2199	2200	2201	2202	2203	2204	2205	2206	2207	2208	2209	2210	2211	2212	2213	2214	2215	2216	2217	2218	2219	2220	2221	2222	2223	2224	2225	2226	2227	2228	2229	2230	2231	2232	2233	2234	2235	2236	2237	2238	2239	2240	2241	2242	2243	2244	2245	2246	2247	2248	2249	2250	2251	2252	2253	2254	2255	2256	2257	2258	2259	2260	2261	2262	2263	2264	2265	2266	2267	2268	2269	2270	2271	2272	2273	2274	2275	2276	2277	2278	2279	2280	2281	2282	2283	2284	2285	2286	2287	2288	2289	2290	2291	2292	2293	2294	2295	2296	2297	2298	2299	2300	2301	2302	2303	2304	2305	2306	2307	2308	2309	2310	2311	2312	2313	2314	2315	2316	2317	2318	2319	2320	2321	2322	2323	2324	2325	2326	2327	2328	2329	2330	2331	2332	2333	2334	2335	2336	2337	2338	2339	2340	2341	2342	2343	2344	2345	2346	2347	2348	2349	2350	2351	2352	2353	2354	2355	2356	2357	2358	2359	2360	2361	2362	2363	2364	2365	2366	2367	2368	2369	2370	2371	2372	2373	2374	2375	2376	2377	2378	2379	2380	2381	2382	2383	2384	2385	2386	2387	2388	2389	2390	2391	2392	2393	2394	2395	2396	2397	2398	2399	2400	2401	2402	2403	2404	2405	2406	2407	2408	2409	2410	2411	2412	2413	2414	2415	2416	2417	2418	2419	2420	2421	2422	2
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The shear stress and thermal energy flux relationships are identical to the results Kliegel (1990) obtained by an Enskog-type solution of the Boltzmann equation.



# Equation Set 10

## Time Average Directional Thermal Energy (10-1)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1^2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{c_1^2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{c_1^2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{c_1^2}] \\
 & + 2 \left[ \overline{c_1^2} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{c_1 c_2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{c_1 c_3} \frac{\partial \overline{u_1}}{\partial x_3} \right] \\
 & + \frac{1}{\rho} \left[ \frac{\partial}{\partial x_1} [\overline{\rho c_1 c_1^2}] + \frac{\partial}{\partial x_2} [\overline{\rho c_2 c_1^2}] + \frac{\partial}{\partial x_3} [\overline{\rho c_3 c_1^2}] \right] \\
 & = \overline{v} \left[ \frac{1}{3} \overline{c^2} - \overline{c_1^2} \right]
 \end{aligned}$$

## Time Average Thermal Shear (10-2)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1 c_2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_2}] \\
 & + \overline{c_1 c_2} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{c_2^2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{c_2 c_3} \frac{\partial \overline{u_1}}{\partial x_3} \\
 & + \overline{c_1^2} \frac{\partial \overline{u_2}}{\partial x_1} + \overline{c_1 c_2} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{c_1 c_3} \frac{\partial \overline{u_2}}{\partial x_3} \\
 & + \frac{1}{\rho} \left[ \frac{\partial}{\partial x_1} [\overline{\rho c_2 c_1^2}] + \frac{\partial}{\partial x_2} [\overline{\rho c_1 c_2^2}] + \frac{\partial}{\partial x_3} [\overline{\rho c_1 c_2 c_3}] \right] \\
 & = -\overline{v c_1 c_2}
 \end{aligned}$$



[illegible]

### Time Average Directional Thermal Energy Fluxes (10-3)

$$\begin{aligned} & \frac{\partial}{\partial t} [\overline{c_1 c_1^2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{c_1 c_1^2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{c_1 c_1^2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_1^2}] \\ & + 3 \left[ \overline{c_1 c_1^2} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{c_2 c_1^2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{c_3 c_1^2} \frac{\partial \overline{u_1}}{\partial x_3} \right] \\ & + 3 \left[ \overline{c_1^2} \frac{\partial}{\partial x_1} [\overline{c_1^2}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{c_1^2}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{c_1^2}] \right] \\ & = \frac{\overline{v}}{2} [\overline{c_1 c^2} - 3 \overline{c_1 c_1^2}] \end{aligned}$$



# Equation Set 10

## Time Average Directional Thermal Energy Fluxes (10-4)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1 c_2^2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{c_1 c_2^2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{c_1 c_2^2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_2^2}] \\
 & + \overline{c_1 c_2^2} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{c_2 c_2^2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{c_3 c_2^2} \frac{\partial \overline{u_1}}{\partial x_3} \\
 & + 2 \left[ \overline{c_2 c_1^2} \frac{\partial \overline{u_2}}{\partial x_1} + \overline{c_1 c_2^2} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{c_1 c_2 c_3} \frac{\partial \overline{u_2}}{\partial x_3} \right] \\
 & + 2 \left[ \overline{c_1 c_2} \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \overline{c_2^2} \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \overline{c_2 c_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_2}] \right] \\
 & + \overline{c_1^2} \frac{\partial}{\partial x_1} [\overline{c_2^2}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{c_2^2}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{c_2^2}] \\
 & = \frac{\overline{v}}{2} \left[ \frac{1}{3} \overline{c_2 c^2} - 3 \overline{c_2 c_1^2} \right]
 \end{aligned}$$



# Equation Set 10

## Time Average Thermal Structure (10-5)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1 c_2 c_3}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{c_1 c_2 c_3}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{c_1 c_2 c_3}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_2 c_3}] \\
 & + \overline{c_1 c_2 c_3} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{c_3 c_2^2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{c_2 c_3^2} \frac{\partial \overline{u_1}}{\partial x_3} \\
 & + \overline{c_3 c_1^2} \frac{\partial \overline{u_2}}{\partial x_1} + \overline{c_1 c_2 c_3} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{c_1 c_3^2} \frac{\partial \overline{u_2}}{\partial x_3} \\
 & + \overline{c_2 c_1^2} \frac{\partial \overline{u_3}}{\partial x_1} + \overline{c_1 c_2^2} \frac{\partial \overline{u_3}}{\partial x_2} + \overline{c_1 c_2 c_3} \frac{\partial \overline{u_3}}{\partial x_3} \\
 & + \overline{c_1^2} \frac{\partial}{\partial x_1} [\overline{c_2 c_3}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{c_2 c_3}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{c_2 c_3}] \\
 & + \overline{c_1 c_2} \frac{\partial}{\partial x_1} [\overline{c_1 c_3}] + \overline{c_2^2} \frac{\partial}{\partial x_2} [\overline{c_1 c_3}] + \overline{c_2 c_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_3}] \\
 & + \overline{c_1 c_3} \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \overline{c_2 c_3} \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \overline{c_3^2} \frac{\partial}{\partial x_3} [\overline{c_1 c_2}] \\
 & = -\frac{3}{2} \overline{v} \overline{c_1 c_2 c_3}
 \end{aligned}$$



Continuity (11-1)

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_1} [\bar{\rho} \bar{u}_1] + \frac{\partial}{\partial x_2} [\bar{\rho} \bar{u}_2] + \frac{\partial}{\partial x_3} [\bar{\rho} \bar{u}_3] = 0$$

Momentum (11-2)

$$\begin{aligned} & \frac{\partial}{\partial t} [\bar{\rho} \bar{u}_1] + \frac{\partial}{\partial x_1} \left[ \bar{\rho} \left[ \bar{u}_1^2 + \overline{u_1'^2} + \overline{c_1^2} \right] \right] \\ & + \frac{\partial}{\partial x_2} \left[ \bar{\rho} \left[ \bar{u}_1 \bar{u}_2 + \overline{u_1' u_2'} + \overline{c_1 c_2} \right] \right] \\ & + \frac{\partial}{\partial x_3} \left[ \bar{\rho} \left[ \bar{u}_1 \bar{u}_3 + \overline{u_1' u_3'} + \overline{c_1 c_3} \right] \right] \\ & = 0 \end{aligned}$$

Turbulent Directional Energy (11-3)

$$\begin{aligned} & \frac{\partial}{\partial t} \left[ \bar{\rho} \left[ \bar{u}_1^2 + \overline{u_1'^2} + \overline{c_1^2} \right] \right] + \frac{\partial}{\partial x_1} \left[ \bar{\rho} \left[ \bar{u}_1^3 + 3 \bar{u}_1 \overline{u_1'^2} + \overline{u_1' u_1'^2} + 3 \bar{u}_1 \overline{c_1^2} + \overline{c_1 c_1^2} \right] \right] \\ & + \frac{\partial}{\partial x_2} \left[ \bar{\rho} \left[ \bar{u}_2 \bar{u}_1^2 + \bar{u}_2 \overline{u_1'^2} + 2 \bar{u}_1 \overline{u_1' u_2'} + \overline{u_2' u_1'^2} + \bar{u}_2 \overline{c_1^2} + 2 \bar{u}_1 \overline{c_1 c_2} + \overline{c_2 c_1^2} \right] \right] \\ & + \frac{\partial}{\partial x_3} \left[ \bar{\rho} \left[ \bar{u}_3 \bar{u}_1^2 + \bar{u}_3 \overline{u_1'^2} + 2 \bar{u}_1 \overline{u_1' u_3'} + \overline{u_3' u_1'^2} + \bar{u}_3 \overline{c_1^2} + 2 \bar{u}_1 \overline{c_1 c_3} + \overline{c_3 c_1^2} \right] \right] \\ & = \bar{\rho} \bar{v} \left[ \frac{\overline{c^2}}{3} - \overline{c_1^2} \right] \end{aligned}$$



## Turbulent Shear (11-4)

$$\begin{aligned}
& \frac{\partial}{\partial t} \left[ \bar{\rho} \left[ \overline{u_1 u_2} + \overline{u'_1 u'_2} + \overline{c_1 c_2} \right] \right] + \frac{\partial}{\partial x_1} \left[ \bar{\rho} \left[ \overline{u_2 u_1^2} + \overline{u_2 u_1'^2} + \overline{u_2' u_1'^2} + 2 \overline{u_1 u_1' u_2'} + \overline{u_2 c_1^2} + 2 \overline{u_1 c_1 c_2} \right] \right] \\
& + \frac{\partial}{\partial x_2} \left[ \bar{\rho} \left[ \overline{u_1 u_2^2} + \overline{u_1 u_2'^2} + 2 \overline{u_2 u_1' u_2'} + \overline{u_1' u_2'^2} + 2 \overline{u_2 c_1 c_2} + \overline{u_1 c_2^2} + \overline{c_1 c_2^2} \right] \right] \\
& + \frac{\partial}{\partial x_3} \left[ \bar{\rho} \left[ \overline{u_1 u_2 u_3} + \overline{u_3 u_1' u_2'} + \overline{u_2 u_1' u_3'} + \overline{u_1 u_2' u_3'} + \overline{u_1' u_2' u_3'} + \overline{u_3 c_1 c_2} + \overline{u_2 c_1 c_3} + \overline{u_1 c_2 c_3} + \overline{c_1 c_2 c_3} \right] \right] \\
& = - \bar{\rho} \bar{v} \overline{c_1 c_2}
\end{aligned}$$

## Turbulent Directional Energy Fluxes (11-5)

$$\begin{aligned}
& \frac{\partial}{\partial t} \left[ \bar{\rho} \left[ \overline{u_1^3} + 3 \overline{u_1 u_1'^2} + \overline{u_1' u_1'^2} + 3 \overline{u_1 c_1^2} + \overline{c_1 c_1^2} \right] \right] \\
& + \frac{\partial}{\partial x_1} \left[ \bar{\rho} \left[ \overline{u_1^4} + 6 \overline{u_1^2 u_1'^2} + 4 \overline{u_1 u_1' u_1'^2} + \overline{u_1'^2 u_1'^2} \right] \right] \\
& + \frac{\partial}{\partial x_1} \left[ \bar{\rho} \left[ 6 \overline{u_1^2 c_1^2} + 4 \overline{u_1 c_1 c_1^2} + 6 \overline{u_1'^2 c_1^2} + \overline{c_1^2 c_1^2} \right] \right] \\
& + \frac{\partial}{\partial x_2} \left[ \bar{\rho} \left[ \overline{u_1^3 u_2} + 3 \overline{u_1^2 u_1' u_2'} + 3 \overline{u_1 u_2 u_1'^2} + \overline{u_2 u_1' u_1'^2} + 3 \overline{u_1 u_2' u_1'^2} + \overline{u_1' u_2' u_1'^2} \right] \right] \\
& + \frac{\partial}{\partial x_2} \left[ \bar{\rho} \left[ 3 \overline{u_1^2 c_1 c_2} + 3 \overline{u_1 u_2 c_1^2} + \overline{u_2 c_1 c_1^2} + 3 \overline{u_1 c_2 c_1^2} + 3 \overline{u_1'^2 c_1 c_2} + 3 \overline{u_1' u_2' c_1^2} + \overline{c_1 c_2 c_1^2} \right] \right] \\
& + \frac{\partial}{\partial x_3} \left[ \bar{\rho} \left[ \overline{u_1^3 u_3} + 3 \overline{u_1^2 u_1' u_3'} + 3 \overline{u_1 u_3 u_1'^2} + \overline{u_3 u_1' u_1'^2} + 3 \overline{u_1 u_3' u_1'^2} + \overline{u_1' u_3' u_1'^2} \right] \right] \\
& + \frac{\partial}{\partial x_3} \left[ \bar{\rho} \left[ 3 \overline{u_1^2 c_1 c_3} + 3 \overline{u_1 u_3 c_1^2} + \overline{u_3 c_1 c_1^2} + 3 \overline{u_1 c_3 c_1^2} + 3 \overline{u_1'^2 c_1 c_3} + 3 \overline{u_1' u_3' c_1^2} + \overline{c_1 c_3 c_1^2} \right] \right] \\
& = \bar{\rho} \bar{v} \left[ 3 \overline{u_1 \left[ \frac{c_1^2}{3} - \overline{c_1^2} \right]} + \frac{1}{2} \left[ \overline{c_1 c_1^2} - 3 \overline{c_1 c_1^2} \right] \right]
\end{aligned}$$



$$\begin{aligned}
 & \frac{\partial}{\partial t} \left[ \bar{\rho} \left[ \overline{u_1 u_2^2} + \overline{u_1 u_2'^2} + 2 \overline{u_2 u_1' u_2'} + \overline{u_1' u_2'^2} + \overline{u_1 c_2^2} + 2 \overline{u_2 c_1 c_2} + \overline{c_1 c_2^2} \right] \right] + \\
 & + \frac{\partial}{\partial x_1} \left[ \bar{\rho} \left[ \overline{u_1^2 u_2^2} + \overline{u_1^2 u_2'^2} + \overline{u_2^2 u_1'^2} + 4 \overline{u_1 u_2 u_1' u_2'} + 2 \overline{u_1 u_1' u_2'^2} + 2 \overline{u_2 u_2' u_1'^2} + \overline{u_1'^2 u_2'^2} \right] \right] \\
 & + \frac{\partial}{\partial x_1} \left[ \bar{\rho} \left[ \overline{u_1^2 c_2^2} + \overline{u_2^2 c_1^2} + 4 \overline{u_1 u_2 c_1 c_2} + 2 \overline{u_1 c_1 c_2^2} + \overline{u_1'^2 c_2^2} + 2 \overline{u_2 c_2 c_1^2} + \overline{u_2'^2 c_1^2} + 4 \overline{u_1' u_2' c_1 c_2} + \overline{c_1^2 c_2^2} \right] \right] \\
 & + \frac{\partial}{\partial x_2} \left[ \bar{\rho} \left[ \overline{u_1 u_2^3} + 3 \overline{u_2^2 u_1' u_2'} + 3 \overline{u_1 u_2 u_2'^2} + \overline{u_1 u_2' u_2'^2} + 3 \overline{u_2 u_1' u_2'^2} + \overline{u_1' u_2' u_2'^2} \right] \right] \\
 & + \frac{\partial}{\partial x_2} \left[ \bar{\rho} \left[ 3 \overline{u_2^2 c_1 c_2} + 3 \overline{u_1 u_2 c_2^2} + \overline{u_1 c_2 c_2^2} + 3 \overline{u_2 c_1 c_2^2} + 3 \overline{u_2'^2 c_1 c_2} + 3 \overline{u_1' u_2' c_2^2} + \overline{c_1 c_2 c_2^2} \right] \right] \\
 & + \frac{\partial}{\partial x_3} \left[ \bar{\rho} \left[ \overline{u_1 u_3 u_2^2} + \overline{u_1 u_3 u_2'^2} + \overline{u_2^2 u_1' u_3} + 2 \overline{u_1 u_2 u_2' u_3} + 2 \overline{u_2 u_3 u_1' u_2'} + 2 \overline{u_2 u_1' u_2' u_3} + \overline{u_1 u_3 u_2'^2} + \overline{u_3 u_1' u_2'^2} + \overline{u_1' u_3 u_2'^2} \right] \right] \\
 & + \frac{\partial}{\partial x_3} \left[ \bar{\rho} \left[ \overline{u_2^2 c_1 c_3} + \overline{u_1 u_3 c_2^2} + 2 \overline{u_1 u_2 c_2 c_3} + 2 \overline{u_2 u_3 c_1 c_2} + \overline{u_1 c_3 c_2^2} + 2 \overline{u_2 c_1 c_2 c_3} + \overline{u_3 c_1 c_2^2} \right] \right] \\
 & + \frac{\partial}{\partial x_3} \left[ \bar{\rho} \left[ \overline{u_2'^2 c_1 c_3} + 2 \overline{u_1' u_2' c_2 c_3} + \overline{u_1' u_3 c_2^2} + 2 \overline{u_2' u_3 c_1 c_2} + \overline{c_1 c_3 c_2^2} \right] \right] \\
 & = \bar{\rho} \bar{v} \left[ \overline{u_1} \left[ \frac{\overline{c_2^2}}{3} - \overline{c_1^2} \right] - 2 \overline{u_2 c_1 c_2} + \frac{1}{6} \left[ \overline{c_1 c_2^2} - 9 \overline{c_1 c_2^2} \right] \right]
 \end{aligned}$$



$$\begin{aligned}
& \frac{\partial}{\partial t} \left[ \bar{\rho} \left[ \overline{u_1 u_2 u_3} + \overline{u_3 c_1 c_2} + \overline{u_2 c_1 c_3} + \overline{u_1 c_2 c_3} + \overline{c_1 c_2 c_3} + \overline{u_3 u'_1 u'_2} + \overline{u_2 u'_1 u'_3} + \overline{u_1 u'_2 u'_3} + \overline{u'_1 u'_2 u'_3} \right] \right] \\
& + \frac{\partial}{\partial x_1} \left[ \bar{\rho} \left[ \overline{u_2 u_3 u_1^2} + \overline{u_1^2 u'_2 u'_3} + 2 \overline{u_1 u_2 u'_1 u'_3} + 2 \overline{u_1 u_3 u'_1 u'_2} + \overline{u_2 u_3 u_1'^2} + 2 \overline{u_1 u'_1 u'_2 u'_3} + \overline{u_2 u'_3 u_1'^2} + \overline{u_3 u'_2 u_1'^2} \right] \right] \\
& + \frac{\partial}{\partial x_1} \left[ \bar{\rho} \left[ \overline{u'_2 u'_3 u_1'^2} + \overline{u_1^2 c_2 c_3} + 2 \overline{u_1 u_3 c_1 c_2} + \overline{u_2 u_3 c_1^2} + 2 \overline{u_1 u_2 c_1 c_3} + 2 \overline{u_1 c_1 c_2 c_3} + \overline{u_2 c_3 c_1^2} + \overline{u_3 c_2 c_1^2} \right] \right] \\
& + \frac{\partial}{\partial x_1} \left[ \bar{\rho} \left[ \overline{u_1'^2 c_2 c_3} + 2 \overline{u'_1 u'_2 c_1 c_3} + 2 \overline{u'_1 u'_3 c_1 c_2} + \overline{u'_2 u'_3 c_1^2} + \overline{c_2 c_3 c_1^2} \right] \right] \\
& + \frac{\partial}{\partial x_2} \left[ \bar{\rho} \left[ \overline{u_1 u_3 u_2^2} + \overline{u_2^2 u'_1 u'_3} + 2 \overline{u_1 u_2 u'_2 u'_3} + 2 \overline{u_2 u_3 u'_1 u'_2} + \overline{u_1 u_3 u_2'^2} + 2 \overline{u_2 u'_1 u'_2 u'_3} + \overline{u_1 u'_3 u_2'^2} + \overline{u_3 u'_1 u_2'^2} \right] \right] \\
& + \frac{\partial}{\partial x_2} \left[ \bar{\rho} \left[ \overline{u'_1 u'_3 u_2'^2} + \overline{u_2^2 c_1 c_3} + 2 \overline{u_2 u_3 c_1 c_2} + \overline{u_1 u_3 c_2^2} + 2 \overline{u_1 u_2 c_2 c_3} + 2 \overline{u_2 c_1 c_2 c_3} + \overline{u_1 c_3 c_2^2} + \overline{u_3 c_1 c_2^2} \right] \right] \\
& + \frac{\partial}{\partial x_2} \left[ \bar{\rho} \left[ \overline{u_2'^2 c_1 c_3} + 2 \overline{u'_1 u'_2 c_2 c_3} + 2 \overline{u'_2 u'_3 c_1 c_2} + \overline{u'_1 u'_3 c_2^2} + \overline{c_1 c_3 c_2^2} \right] \right] \\
& + \frac{\partial}{\partial x_3} \left[ \bar{\rho} \left[ \overline{u_1 u_2 u_3^2} + \overline{u_3^2 u'_1 u'_2} + 2 \overline{u_2 u_3 u'_1 u'_3} + 2 \overline{u_1 u_3 u'_2 u'_3} + \overline{u_1 u_2 u_3'^2} + 2 \overline{u_3 u'_1 u'_2 u'_3} + \overline{u_2 u'_1 u_3'^2} + \overline{u_1 u'_2 u_3'^2} \right] \right] \\
& + \frac{\partial}{\partial x_3} \left[ \bar{\rho} \left[ \overline{u'_1 u'_2 u_3'^2} + \overline{u_3^2 c_1 c_2} + 2 \overline{u_1 u_3 c_2 c_3} + \overline{u_1 u_2 c_3^2} + 2 \overline{u_2 u_3 c_1 c_3} + 2 \overline{u_3 c_1 c_2 c_3} + \overline{u_2 c_1 c_3^2} + \overline{u_1 c_2 c_3^2} \right] \right] \\
& + \frac{\partial}{\partial x_3} \left[ \bar{\rho} \left[ \overline{u_3'^2 c_1 c_2} + 2 \overline{u'_2 u'_3 c_1 c_3} + 2 \overline{u'_1 u'_3 c_2 c_3} + \overline{u'_1 u'_2 c_3^2} + \overline{c_1 c_2 c_3^2} \right] \right] \\
& = -\bar{\rho} \bar{v} \left[ \overline{u_1 c_2 c_3} + \overline{u_2 c_1 c_3} + \overline{u_3 c_1 c_2} + \frac{3}{2} \overline{c_1 c_2 c_3} \right]
\end{aligned}$$



## Equation Set 12

### Directional Turbulent Energy (12-1)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{u_1'^2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{u_1'^2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{u_1'^2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{u_1'^2}] \\
 & + 2 \left[ \overline{u_1'^2} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{u_1' u_2'} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{u_1' u_3'} \frac{\partial \overline{u_1}}{\partial x_3} \right] \\
 & + \frac{1}{\rho} \left[ \frac{\partial}{\partial x_1} [\overline{\rho u_1' u_1'^2}] + \frac{\partial}{\partial x_2} [\overline{\rho u_2' u_1'^2}] + \frac{\partial}{\partial x_3} [\overline{\rho u_3' u_1'^2}] \right] \\
 & = 0
 \end{aligned}$$

### Turbulent Shear (12-2)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{u_1' u_2'}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{u_1' u_2'}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{u_1' u_2'}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{u_1' u_2'}] \\
 & + \overline{u_1' u_2'} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{u_2'^2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{u_2' u_3'} \frac{\partial \overline{u_1}}{\partial x_3} \\
 & + \overline{u_1'^2} \frac{\partial \overline{u_2}}{\partial x_1} + \overline{u_1' u_2'} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{u_1' u_3'} \frac{\partial \overline{u_2}}{\partial x_3} \\
 & + \frac{1}{\rho} \left[ \frac{\partial}{\partial x_1} [\overline{\rho u_2' u_1'^2}] + \frac{\partial}{\partial x_2} [\overline{\rho u_1' u_2'^2}] + \frac{\partial}{\partial x_3} [\overline{\rho u_1' u_2' u_3'}] \right] \\
 & = 0
 \end{aligned}$$



## Directional Turbulent Energy Fluxes (12-3)

$$\begin{aligned}
& \frac{\partial}{\partial t} [\overline{u'_1 u'^2_1}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{u'_1 u'^2_1}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{u'_1 u'^2_1}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'^2_1}] \\
& + 3 \left[ \overline{u'_1 u'^2_1} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{u'_2 u'^2_1} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{u'_3 u'^2_1} \frac{\partial \overline{u_1}}{\partial x_3} \right] \\
& - 3 \overline{u'^2_1} \left[ \frac{\partial}{\partial x_1} [\overline{u'^2_1}] + \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2}] + \frac{\partial}{\partial x_3} [\overline{u'_1 u'_3}] \right] \\
& + 3 \overline{c^2_1} \left[ \frac{\partial}{\partial x_1} [\overline{u'^2_1}] + \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2}] + \frac{\partial}{\partial x_3} [\overline{u'_1 u'_3}] \right] \\
& + 3 \left[ \overline{u'^2_1} \frac{\partial}{\partial x_1} [\overline{c^2_1}] + \overline{u'_1 u'_2} \frac{\partial}{\partial x_2} [\overline{c^2_1}] + \overline{u'_1 u'_3} \frac{\partial}{\partial x_3} [\overline{c^2_1}] \right] \\
& + \frac{\partial}{\partial x_1} [\overline{u'^2_1 u'^2_1}] + \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2 u'^2_1}] + \frac{\partial}{\partial x_3} [\overline{u'_1 u'_3 u'^2_1}] \\
& + \left[ \overline{u'^2_1 u'^2_1} - 3 \overline{u'^2_1} [\overline{u'^2_1}] \right] \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho}}{\partial x_1} \\
& + \left[ \overline{u'_1 u'_2 u'^2_1} - 3 \overline{u'_1 u'_2} [\overline{u'^2_1}] \right] \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho}}{\partial x_2} \\
& + \left[ \overline{u'_1 u'_3 u'^2_1} - 3 \overline{u'_1 u'_3} [\overline{u'^2_1}] \right] \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho}}{\partial x_3} \\
& = 0
\end{aligned}$$



## Directional Turbulent Energy Fluxes (12-4)

$$\begin{aligned}
& \frac{\partial}{\partial t} [\overline{u'_1 u'^2_2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{u'_1 u'^2_2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{u'_1 u'^2_2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'^2_2}] \\
& + 2 \left[ \overline{u'_1 u'^2_2} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{u'_2 u'^2_1} \frac{\partial \overline{u_2}}{\partial x_1} + \overline{u'_1 u'_2 u'_3} \frac{\partial \overline{u_2}}{\partial x_3} \right] \\
& + \overline{u'_2 u'^2_2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{u'_1 u'^2_1} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{u'_3 u'^2_1} \frac{\partial \overline{u_1}}{\partial x_3} \\
& - 2 \overline{u'_1 u'_2} \left[ \frac{\partial}{\partial x_1} [\overline{u'^2_2}] + \frac{\partial}{\partial x_1} [\overline{u'_1 u'_2}] + \frac{\partial}{\partial x_3} [\overline{u'_2 u'_3}] \right] \\
& - \overline{u'^2_2} \left[ \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2}] + \frac{\partial}{\partial x_1} [\overline{u'^2_1}] + \frac{\partial}{\partial x_3} [\overline{u'_1 u'_3}] \right] \\
& + 2 \left[ \overline{c^2_2} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_2}] + \overline{c_1 c_2} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_2}] + \overline{c_2 c_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'_2}] \right] \\
& + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{u'^2_2}] + \overline{c^2_1} \frac{\partial}{\partial x_1} [\overline{u'^2_2}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{u'^2_2}] \\
& + 2 \left[ \overline{u'^2_2} \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2}] + \overline{u'_1 u'_2} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_2}] + \overline{u'_2 u'_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'_2}] \right] \\
& + \overline{u'_1 u'_2} \frac{\partial}{\partial x_2} [\overline{c^2_2}] + \overline{u'^2_1} \frac{\partial}{\partial x_1} [\overline{c^2_2}] + \overline{u'_1 u'_3} \frac{\partial}{\partial x_3} [\overline{c^2_1}] \\
& + \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2 u'^2_2}] + \frac{\partial}{\partial x_1} [\overline{u'^2_1 u'^2_2}] + \frac{\partial}{\partial x_3} [\overline{u'_1 u'_3 u'^2_2}] \\
& + \left[ \overline{u'_1 u'_2 u'^2_2} - 3 \overline{u'_1 u'_2} [\overline{u'^2_2}] \right] \frac{1}{\rho} \frac{\partial \overline{\rho}}{\partial x_2} \\
& + \left[ \overline{u'^2_1 u'^2_2} - \overline{u'^2_1} [\overline{u'^2_2}] - 2 \overline{u'_1 u'_2} [\overline{u'_1 u'_2}] \right] \frac{1}{\rho} \frac{\partial \overline{\rho}}{\partial x_1} \\
& + \left[ \overline{u'_1 u'_3 u'^2_2} - \overline{u'_1 u'_3} [\overline{u'^2_2}] - 2 \overline{u'_1 u'_2} [\overline{u'_2 u'_3}] \right] \frac{1}{\rho} \frac{\partial \overline{\rho}}{\partial x_3} \\
& = 0
\end{aligned}$$



$$\begin{aligned}
& \frac{\partial}{\partial t} [\overline{u'_1 u'_2 u'_3}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_2 u'_3}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2 u'_3}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'_2 u'_3}] \\
& + \overline{u'_1 u'_2 u'_3} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{u'_3 u'^2_2} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{u'_2 u'^2_3} \frac{\partial \overline{u_1}}{\partial x_3} \\
& + \overline{u'_3 u'^2_1} \frac{\partial \overline{u_2}}{\partial x_1} + \overline{u'_1 u'_2 u'_3} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{u'_1 u'^2_3} \frac{\partial \overline{u_2}}{\partial x_3} \\
& + \overline{u'_2 u'^2_1} \frac{\partial \overline{u_3}}{\partial x_1} + \overline{u'_1 u'^2_2} \frac{\partial \overline{u_2}}{\partial x_3} + \overline{u'_1 u'_2 u'_3} \frac{\partial \overline{u_3}}{\partial x_3} \\
& - \overline{u'_1 u'_2} \left[ \frac{\partial}{\partial x_1} [\overline{u'_1 u'_3}] + \frac{\partial}{\partial x_2} [\overline{u'_2 u'_3}] + \frac{\partial}{\partial x_3} [\overline{u'^2_3}] \right] \\
& - \overline{u'_1 u'_3} \left[ \frac{\partial}{\partial x_1} [\overline{u'_1 u'_2}] + \frac{\partial}{\partial x_2} [\overline{u'^2_2}] + \frac{\partial}{\partial x_3} [\overline{u'_2 u'_3}] \right] \\
& - \overline{u'_2 u'_3} \left[ \frac{\partial}{\partial x_1} [\overline{u'^2_1}] + \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2}] + \frac{\partial}{\partial x_3} [\overline{u'_1 u'_3}] \right] \\
& + \overline{u'^2_1} \frac{\partial}{\partial x_1} [\overline{c_2 c_3}] + \overline{u'_1 u'_2} \frac{\partial}{\partial x_2} [\overline{c_2 c_3}] + \overline{u'_1 u'_3} \frac{\partial}{\partial x_3} [\overline{c_2 c_3}] \\
& + \overline{u'_1 u'_2} \frac{\partial}{\partial x_1} [\overline{c_1 c_3}] + \overline{u'^2_2} \frac{\partial}{\partial x_2} [\overline{c_1 c_3}] + \overline{u'_2 u'_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_3}] \\
& + \overline{u'_1 u'_3} \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \overline{u'_2 u'_3} \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \overline{u'^2_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_2}] \\
& + \overline{c_1^2} \frac{\partial}{\partial x_1} [\overline{u'_2 u'_3}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{u'_2 u'_3}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{u'_2 u'_3}] \\
& + \overline{c_1 c_3} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_2}] + \overline{c_2 c_3} \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2}] + \overline{c_3^2} \frac{\partial}{\partial x_3} [\overline{u'_1 u'_2}] \\
& + \overline{c_1 c_2} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_3}] + \overline{c_2^2} \frac{\partial}{\partial x_2} [\overline{u'_1 u'_3}] + \overline{c_2 c_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'_3}] \\
& + \frac{\partial}{\partial x_1} [\overline{u'_2 u'_3 u'^2_1}] + \frac{\partial}{\partial x_2} [\overline{u'_1 u'_3 u'^2_2}] + \frac{\partial}{\partial x_3} [\overline{u'_1 u'_2 u'^2_3}] \\
& + \left[ \overline{u'_2 u'_3 u'^2_1} - \overline{u'_2 u'_3} [\overline{u'^2_1}] - 2 \overline{u'_1 u'_2} [\overline{u'_1 u'_3}] \right] \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho}}{\partial x_1} \\
& + \left[ \overline{u'_1 u'_3 u'^2_2} - \overline{u'_1 u'_3} [\overline{u'^2_2}] - 2 \overline{u'_1 u'_2} [\overline{u'_2 u'_3}] \right] \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho}}{\partial x_2} \\
& + \left[ \overline{u'_1 u'_2 u'^2_3} - \overline{u'_1 u'_2} [\overline{u'^2_3}] - 2 \overline{u'_1 u'_3} [\overline{u'_2 u'_3}] \right] \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho}}{\partial x_3} \\
& = 0
\end{aligned}$$

Equation Set 12 Turbulent Structure (12-5)



Equation Set 13 (13-1; 13-2; 13-3; 13-4)

$$\overline{c_1^2 c_1^2} = 3 \overline{c_1^2} \left[ \overline{c_1^2} \right]$$

$$\overline{c_1 c_2 c_1^2} = 3 \overline{c_1 c_2} \left[ \overline{c_1^2} \right]$$

$$\overline{c_1^2 c_2^2} = \overline{c_1^2} \left[ \overline{c_2^2} \right] + 2 \overline{c_1 c_2} \left[ \overline{c_1 c_2} \right]$$

$$\overline{c_1 c_2 c_3^2} = \overline{c_1 c_2} \left[ \overline{c_3^2} \right] + 2 \overline{c_1 c_3} \left[ \overline{c_2 c_3} \right]$$

neglected terms being of Burnett order.

Equation Set 14 ( 14-1; 14-2; 14-3; 14-4)

$$\overline{u_1'^2 u_1'^2} = 3 \overline{u_1'^2} \left[ \overline{u_1'^2} \right]$$

$$\overline{u_1' u_2' u_1'^2} = 3 \overline{u_1' u_2'} \left[ \overline{u_1'^2} \right]$$

$$\overline{u_1'^2 u_2'^2} = \overline{u_1'^2} \left[ \overline{u_2'^2} \right] + 2 \overline{u_1' u_2'} \left[ \overline{u_1' u_2'} \right]$$

$$\overline{u_1' u_2' u_3'^2} = \overline{u_1' u_2'} \left[ \overline{u_3'^2} \right] + 2 \overline{u_1' u_3'} \left[ \overline{u_2' u_3'} \right]$$

neglected terms being of Burnett order.



## Equation Set 15

Time Average Continuity (15-1)

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_1} [\bar{\rho} \bar{u}_1] + \frac{\partial}{\partial x_2} [\bar{\rho} \bar{u}_2] + \frac{\partial}{\partial x_3} [\bar{\rho} \bar{u}_3] = 0$$

Time Average Momentum (15-2)

$$\begin{aligned} & \frac{\partial}{\partial t} [\bar{\rho} \bar{u}_1] + \frac{\partial}{\partial x_1} \left[ \bar{\rho} \left[ \bar{u}_1^2 + \overline{u_1'^2} + \overline{c_1^2} \right] \right] \\ & + \frac{\partial}{\partial x_2} \left[ \bar{\rho} \left[ \overline{u_1 u_2} + \overline{u_1' u_2'} + \overline{c_1 c_2} \right] \right] \\ & + \frac{\partial}{\partial x_3} \left[ \bar{\rho} \left[ \overline{u_1 u_3} + \overline{u_1' u_3'} + \overline{c_1 c_3} \right] \right] \\ & = 0 \end{aligned}$$



# Equation Set 15

## Time Average Directional Thermal Energy (15-3)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1^2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{c_1^2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{c_1^2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{c_1^2}] \\
 & + 2 \left[ \overline{c_1^2} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{c_1 c_2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{c_1 c_3} \frac{\partial \overline{u_1}}{\partial x_3} \right] \\
 & + \frac{1}{\rho} \left[ \frac{\partial}{\partial x_1} [\overline{\rho c_1 c_1^2}] + \frac{\partial}{\partial x_2} [\overline{\rho c_2 c_1^2}] + \frac{\partial}{\partial x_3} [\overline{\rho c_3 c_1^2}] \right] \\
 & = \overline{v} \left[ \frac{1}{3} \overline{c^2} - \overline{c_1^2} \right]
 \end{aligned}$$

## Time Average Thermal Shear (15-4)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1 c_2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_2}] \\
 & + \overline{c_1 c_2} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{c_2^2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{c_2 c_3} \frac{\partial \overline{u_1}}{\partial x_3} \\
 & + \overline{c_1^2} \frac{\partial \overline{u_2}}{\partial x_1} + \overline{c_1 c_2} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{c_1 c_3} \frac{\partial \overline{u_2}}{\partial x_3} \\
 & + \frac{1}{\rho} \left[ \frac{\partial}{\partial x_1} [\overline{\rho c_2 c_1^2}] + \frac{\partial}{\partial x_2} [\overline{\rho c_1 c_2^2}] + \frac{\partial}{\partial x_3} [\overline{\rho c_1 c_2 c_3}] \right] \\
 & = -\overline{v c_1 c_2}
 \end{aligned}$$



# Equation Set 15

## Closed Time Average Directional Thermal Energy Fluxes (15-5)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1 c_1^2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{c_1 c_1^2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{c_1 c_1^2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_1^2}] \\
 & + 3 \left[ \overline{c_1 c_1^2} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{c_2 c_1^2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{c_3 c_1^2} \frac{\partial \overline{u_1}}{\partial x_3} \right] \\
 & + 3 \left[ \overline{c_1^2} \frac{\partial}{\partial x_1} [\overline{c_1^2}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{c_1^2}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{c_1^2}] \right] \\
 & = \frac{\overline{v}}{2} [\overline{c_1 c^2} - 3 \overline{c_1 c_1^2}]
 \end{aligned}$$

(15-6)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1 c_2^2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{c_1 c_2^2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{c_1 c_2^2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_2^2}] \\
 & + \overline{c_1 c_2^2} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{c_2 c_2^2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{c_3 c_2^2} \frac{\partial \overline{u_1}}{\partial x_3} \\
 & + 2 \left[ \overline{c_2 c_1^2} \frac{\partial \overline{u_2}}{\partial x_1} + \overline{c_1 c_2^2} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{c_1 c_2 c_3} \frac{\partial \overline{u_2}}{\partial x_3} \right] \\
 & + 2 \left[ \overline{c_1 c_2} \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \overline{c_2^2} \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \overline{c_2 c_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_2}] \right] \\
 & + \overline{c_1^2} \frac{\partial}{\partial x_1} [\overline{c_2^2}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{c_2^2}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{c_2^2}] \\
 & = \frac{\overline{v}}{2} \left[ \frac{1}{3} \overline{c_2 c^2} - 3 \overline{c_2 c_1^2} \right]
 \end{aligned}$$



Closed Time Average Thermal Structure (15-7)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{c_1 c_2 c_3}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{c_1 c_2 c_3}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{c_1 c_2 c_3}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_2 c_3}] \\
 & + \overline{c_1 c_2 c_3} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{c_3 c_2^2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{c_2 c_3^2} \frac{\partial \overline{u_1}}{\partial x_3} \\
 & + \overline{c_3 c_1^2} \frac{\partial \overline{u_2}}{\partial x_1} + \overline{c_1 c_2 c_3} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{c_1 c_3^2} \frac{\partial \overline{u_2}}{\partial x_3} \\
 & + \overline{c_2 c_1^2} \frac{\partial \overline{u_3}}{\partial x_1} + \overline{c_1 c_2^2} \frac{\partial \overline{u_3}}{\partial x_2} + \overline{c_1 c_2 c_3} \frac{\partial \overline{u_3}}{\partial x_3} \\
 & + \overline{c_1^2} \frac{\partial}{\partial x_1} [\overline{c_2 c_3}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{c_2 c_3}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{c_2 c_3}] \\
 & + \overline{c_1 c_2} \frac{\partial}{\partial x_1} [\overline{c_1 c_3}] + \overline{c_2^2} \frac{\partial}{\partial x_2} [\overline{c_1 c_3}] + \overline{c_2 c_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_3}] \\
 & + \overline{c_1 c_3} \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \overline{c_2 c_3} \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \overline{c_3^2} \frac{\partial}{\partial x_3} [\overline{c_1 c_2}] \\
 & = -\frac{3}{2} \overline{v} \overline{c_1 c_2 c_3}
 \end{aligned}$$



Time Average Directional Turbulent Energy (15-8)

$$\begin{aligned}
& \frac{\partial}{\partial t} [\overline{u_1'^2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{u_1'^2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{u_1'^2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{u_1'^2}] \\
& + 2 \left[ \overline{u_1'^2} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{u_1' u_2'} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{u_1' u_3'} \frac{\partial \overline{u_1}}{\partial x_3} \right] \\
& + \frac{1}{\rho} \left[ \frac{\partial}{\partial x_1} [\overline{\rho u_1' u_1'^2}] + \frac{\partial}{\partial x_2} [\overline{\rho u_2' u_1'^2}] + \frac{\partial}{\partial x_3} [\overline{\rho u_3' u_1'^2}] \right] \\
& = 0
\end{aligned}$$

Time Average Turbulent Shear (15-9)

$$\begin{aligned}
& \frac{\partial}{\partial t} [\overline{u_1' u_2'}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{u_1' u_2'}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{u_1' u_2'}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{u_1' u_2'}] \\
& + \overline{u_1' u_2'} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{u_2'^2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{u_2' u_3'} \frac{\partial \overline{u_1}}{\partial x_3} \\
& + \overline{u_1'^2} \frac{\partial \overline{u_2}}{\partial x_1} + \overline{u_1' u_2'} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{u_1' u_3'} \frac{\partial \overline{u_2}}{\partial x_3} \\
& + \frac{1}{\rho} \left[ \frac{\partial}{\partial x_1} [\overline{\rho u_2' u_1'^2}] + \frac{\partial}{\partial x_2} [\overline{\rho u_1' u_2'^2}] + \frac{\partial}{\partial x_3} [\overline{\rho u_1' u_2' u_3'}] \right] \\
& = 0
\end{aligned}$$



# Equation Set 15

Closed Time Average Turbulent Directional Energy Fluxes (15-10)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{u'_1 u'^2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{u'_1 u'^2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{u'_1 u'^2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'^2}] \\
 & + 3 \left[ \overline{u'_1 u'^2} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{u'_2 u'^2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{u'_3 u'^2} \frac{\partial \overline{u_1}}{\partial x_3} \right] \\
 & + 3 \left[ \overline{u_1'^2} \frac{\partial}{\partial x_1} [\overline{u_1'^2}] + \overline{u_1' u'_2} \frac{\partial}{\partial x_2} [\overline{u_1'^2}] + \overline{u_1' u'_3} \frac{\partial}{\partial x_3} [\overline{u_1'^2}] \right] \\
 & + 3 \overline{c_1^2} \left[ \frac{\partial}{\partial x_1} [\overline{u_1'^2}] + \frac{\partial}{\partial x_2} [\overline{u_1' u'_2}] + \frac{\partial}{\partial x_3} [\overline{u_1' u'_3}] \right] \\
 & + 3 \left[ \overline{u_1'^2} \frac{\partial}{\partial x_1} [\overline{c_1^2}] + \overline{u_1' u'_2} \frac{\partial}{\partial x_2} [\overline{c_1^2}] + \overline{u_1' u'_3} \frac{\partial}{\partial x_3} [\overline{c_1^2}] \right] \\
 & = 0
 \end{aligned}$$



# Equation Set 15

Closed Time Average Turbulent Directional Energy Fluxes (15-11)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{u'_1 u'^2_2}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{u'_1 u'^2_2}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{u'_1 u'^2_2}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'^2_2}] \\
 & + 2 \left[ \overline{u'_2 u'^2_1} \frac{\partial \overline{u_2}}{\partial x_1} + \overline{u'_1 u'^2_2} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{u'_1 u'_2 u'_3} \frac{\partial \overline{u_2}}{\partial x_3} \right] \\
 & + \overline{u'_1 u'^2_1} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{u'_2 u'^2_2} \frac{\partial \overline{u_1}}{\partial x_2} + \overline{u'_3 u'^2_1} \frac{\partial \overline{u_1}}{\partial x_3} \\
 & + 2 \left[ \overline{u'_1 u'_2} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_2}] + \overline{u'^2_2} \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2}] + \overline{u'_2 u'_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'_2}] \right] \\
 & + \overline{u'^2_1} \frac{\partial}{\partial x_1} [\overline{u'^2_2}] + \overline{u'_1 u'_2} \frac{\partial}{\partial x_2} [\overline{u'^2_2}] + \overline{u'_1 u'_3} \frac{\partial}{\partial x_3} [\overline{u'^2_2}] \\
 & + 2 \left[ \overline{u'_1 u'_2} \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \overline{u'^2_2} \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \overline{u'_2 u'_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_2}] \right] \\
 & + \overline{u'^2_1} \frac{\partial}{\partial x_1} [\overline{c^2_2}] + \overline{u'_1 u'_2} \frac{\partial}{\partial x_2} [\overline{c^2_2}] + \overline{u'_1 u'_3} \frac{\partial}{\partial x_3} [\overline{c^2_2}] \\
 & + 2 \left[ \overline{c_1 c_2} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_2}] + \overline{c^2_2} \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2}] + \overline{c_2 c_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'_2}] \right] \\
 & + \overline{c^2_1} \frac{\partial}{\partial x_1} [\overline{u'^2_2}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{u'^2_2}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{u'^2_2}] \\
 & = 0
 \end{aligned}$$



Equation Set 15

Closed Time Average Turbulent Structure (15-12)

$$\begin{aligned}
 & \frac{\partial}{\partial t} [\overline{u'_1 u'_2 u'_3}] + \overline{u_1} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_2 u'_3}] + \overline{u_2} \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2 u'_3}] + \overline{u_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'_2 u'_3}] \\
 & + \overline{u'_1 u'_2 u'_3} \frac{\partial \overline{u_1}}{\partial x_1} + \overline{u'_1 u'_2 u'_3} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{u'_1 u'_2 u'_3} \frac{\partial \overline{u_3}}{\partial x_3} \\
 & + \overline{u'_3 u_1'^2} \frac{\partial \overline{u_2}}{\partial x_1} + \overline{u'_1 u'_2 u'_3} \frac{\partial \overline{u_2}}{\partial x_2} + \overline{u'_1 u'_3} \frac{\partial \overline{u_2}}{\partial x_3} \\
 & + \overline{u'_2 u_1'^2} \frac{\partial \overline{u_3}}{\partial x_1} + \overline{u'_1 u'_2} \frac{\partial \overline{u_3}}{\partial x_2} + \overline{u'_1 u'_2 u'_3} \frac{\partial \overline{u_3}}{\partial x_3} \\
 & + \overline{u'_1 u'_3} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_2}] + \overline{u'_2 u'_3} \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2}] + \overline{u_3'^2} \frac{\partial}{\partial x_3} [\overline{u'_1 u'_2}] \\
 & + \overline{u'_1 u'_2} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_3}] + \overline{u_2'^2} \frac{\partial}{\partial x_2} [\overline{u'_1 u'_3}] + \overline{u'_2 u'_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'_3}] \\
 & + \overline{u_1'^2} \frac{\partial}{\partial x_1} [\overline{u'_2 u'_3}] + \overline{u'_1 u'_2} \frac{\partial}{\partial x_2} [\overline{u'_2 u'_3}] + \overline{u'_1 u'_3} \frac{\partial}{\partial x_3} [\overline{u'_2 u'_3}] \\
 & + \overline{c_1 c_3} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_2}] + \overline{c_2 c_3} \frac{\partial}{\partial x_2} [\overline{u'_1 u'_2}] + \overline{c_3^2} \frac{\partial}{\partial x_3} [\overline{u'_1 u'_2}] \\
 & + \overline{c_1 c_2} \frac{\partial}{\partial x_1} [\overline{u'_1 u'_3}] + \overline{c_2^2} \frac{\partial}{\partial x_2} [\overline{u'_1 u'_3}] + \overline{c_2 c_3} \frac{\partial}{\partial x_3} [\overline{u'_1 u'_3}] \\
 & + \overline{c_1^2} \frac{\partial}{\partial x_1} [\overline{u'_2 u'_3}] + \overline{c_1 c_2} \frac{\partial}{\partial x_2} [\overline{u'_2 u'_3}] + \overline{c_1 c_3} \frac{\partial}{\partial x_3} [\overline{u'_2 u'_3}] \\
 & + \overline{u'_1 u'_3} \frac{\partial}{\partial x_1} [\overline{c_1 c_2}] + \overline{u'_2 u'_3} \frac{\partial}{\partial x_2} [\overline{c_1 c_2}] + \overline{u_3'^2} \frac{\partial}{\partial x_3} [\overline{c_1 c_2}] \\
 & + \overline{u'_1 u'_2} \frac{\partial}{\partial x_1} [\overline{c_1 c_3}] + \overline{u_2'^2} \frac{\partial}{\partial x_2} [\overline{c_1 c_3}] + \overline{u'_2 u'_3} \frac{\partial}{\partial x_3} [\overline{c_1 c_3}] \\
 & + \overline{u_1'^2} \frac{\partial}{\partial x_1} [\overline{c_2 c_3}] + \overline{u'_1 u'_2} \frac{\partial}{\partial x_2} [\overline{c_2 c_3}] + \overline{u'_1 u'_3} \frac{\partial}{\partial x_3} [\overline{c_2 c_3}] \\
 & = 0
 \end{aligned}$$



# Equation Set 16

(16-1; 16-2; 16-3)

$$\begin{aligned}\overline{c_1^{l+l} c_2^m c_3^n} &= l \left[ \overline{c_1^2} \right] \overline{c_1^{l-l} c_2^m c_3^n} + m \left[ \overline{c_1 c_2} \right] \overline{c_1^l c_2^{m-l} c_3^n} + n \left[ \overline{c_1 c_3} \right] \overline{c_1^l c_2^m c_3^{n-l}} \\ \overline{c_1^l c_2^{m+l} c_3^n} &= l \left[ \overline{c_1 c_2} \right] \overline{c_1^{l-l} c_2^m c_3^n} + m \left[ \overline{c_2^2} \right] \overline{c_1^l c_2^{m-l} c_3^n} + n \left[ \overline{c_2 c_3} \right] \overline{c_1^l c_2^m c_3^{n-l}} \\ \overline{c_1^l c_2^m c_3^{n+l}} &= l \left[ \overline{c_1 c_3} \right] \overline{c_1^{l-l} c_2^m c_3^n} + m \left[ \overline{c_2 c_3} \right] \overline{c_1^l c_2^{m-l} c_3^n} + n \left[ \overline{c_3^2} \right] \overline{c_1^l c_2^m c_3^{n-l}}\end{aligned}$$

neglected terms being of Burnett order.

# Equation Set 17

(17-1; 17-2; 17-3)

$$\begin{aligned}\overline{u_1^{l+l} u_2'^m u_3'^n} &= l \left[ \overline{u_1'^2} \right] \overline{u_1^{l-l} u_2'^m u_3'^n} + m \left[ \overline{u_1' u_2'} \right] \overline{u_1^{l-l} u_2'^{m-l} u_3'^n} + n \left[ \overline{u_1' u_3'} \right] \overline{u_1^{l-l} u_2'^m u_3'^{n-l}} \\ \overline{u_1^{l-l} u_2'^{m+l} u_3'^n} &= l \left[ \overline{u_1' u_2'} \right] \overline{u_1^{l-l} u_2'^m u_3'^n} + m \left[ \overline{u_2'^2} \right] \overline{u_1^{l-l} u_2'^{m-l} u_3'^n} + n \left[ \overline{u_2' u_3'} \right] \overline{u_1^{l-l} u_2'^m u_3'^{n-l}} \\ \overline{u_1^{l-l} u_2'^m u_3'^{n+l}} &= l \left[ \overline{u_1' u_3'} \right] \overline{u_1^{l-l} u_2'^m u_3'^n} + m \left[ \overline{u_2' u_3'} \right] \overline{u_1^{l-l} u_2'^{m-l} u_3'^n} + n \left[ \overline{u_3'^2} \right] \overline{u_1^{l-l} u_2'^m u_3'^{n-l}}\end{aligned}$$

neglected terms being of Burnett order.



Equation 18

(18-1; 18-2; 18-3)

$$\begin{aligned}
 \overline{c_1^{l+l} c_2^m c_3^n}^* &= l \left[ \frac{\overline{c^2}}{3} \right] \overline{c_1^{l-l} c_2^m c_3^n}, \quad l \text{ odd, } m \text{ and } n \text{ even} \\
 &= 0 \quad \quad \quad l, m \text{ and } n \text{ all odd} \\
 \overline{c_1^l c_2^{m+l} c_3^n}^* &= m \left[ \frac{\overline{c^2}}{3} \right] \overline{c_1^l c_2^{m-l} c_3^n}, \quad m \text{ odd, } l \text{ and } n \text{ even} \\
 &= 0 \quad \quad \quad l, m \text{ and } n \text{ all odd} \\
 \overline{c_1^l c_2^m c_3^{n+l}}^* &= n \left[ \frac{\overline{c^2}}{3} \right] \overline{c_1^l c_2^m c_3^{n-l}}, \quad n \text{ odd, } l \text{ and } m \text{ even} \\
 &= 0 \quad \quad \quad l, m \text{ and } n \text{ all odd}
 \end{aligned}$$

neglected terms being of Burnett order, where  $l+m+n = 2N + 1$ , an odd number.